CHROMATICITY AND ADJOINT POLYNOMIALS OF GRAPHS

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PROEFSCHRIFT

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Haixing Zhao
May 2005, Enschede

Preface

This thesis is the result of almost three years of research in the field of chromaticity of graphs between September 2002 and March 2005. After an introductory chapter the reader will find five chapters that contain two main parts within this research field. The two parts have strong connections with each other. The first part, Chapters 2 and 3, is on algebraic properties and roots of adjoint polynomials. The second part, Chapters 4, 5 and 6, studies mainly the chromaticity of some classes of graphs, that is dense graphs, complete multipartite graphs and general multipartite graphs. Some results of this thesis have been published in journals: see the following list.

1. On the minimum real roots of the σ-polynomials and chromatic uniqueness of graphs, *Discrete Mathematics* **281** (2004), 277-294 (with X. Li, R. Liu and S. Zhang).
2. A complete solution to a conjecture on chromatic uniqueness of complete tripartite graphs, *Discrete Mathematics* **289** (2004), 175-179 (with R. Liu and C. Ye).
3. On problems and conjectures on adjointly equivalent graphs, *Discrete Mathematics* **295** (2005), 203-212 (with X. Li and R. Liu).
4. The chromaticity of certain complete multipartite graphs, *Graphs and Combinatorics* **20** (2004), 423-434 (with X. Li, R. Liu and C. Ye).
5. On properties of adjoint polynomials of graphs and their applications, *Australasian Journal of Combinatorics* **30** (2004), 291-307 (with X. Li,

R. Liu and L. Wang).

6. A note on adjoint polynomials and uniquely colorable graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **45**(2003), 123-128 (with R. Liu).

Contents

Chapter 1

Introduction

For the purpose of tackling the four-color problem, Birkhoff in 1912 [2] introduced the chromatic polynomial of a map M, denoted by $P(M, \lambda)$, which is the number of proper λ-colorings of a map M. If one could prove that $P(M, 4) > 0$ for all maps M, then this would give a positive answer to the four-color problem. Later, Birkhoff and Lewis [3] obtained some results concerning the distribution of real roots of chromatic polynomials of planar graphs and conjectured that these polynomials have no real roots greater than or equal to four. The conjecture remains open.

In 1932, Whitney [76] generalized the notion of a chromatic polynomial to that of an arbitrary graph and gave many fundamental results for its chromatic polynomial. In 1968, Read [65] asked whether it is possible to find a set of necessary and sufficient algebraic conditions for a polynomial to be the chromatic polynomial of some graph. In particular, Read asked for a necessary and sufficient condition for two graphs to be chromatically equivalent; that is, to have the same chromatic polynomial.

In 1978, Chao and Whitehead [8] defined a graph to be chromatically unique if no other graphs share its chromatic polynomial. They gave several families of chromatically unique graphs [9, 10, 11]. Since then many researchers have been studying chromatic uniqueness and chromatic equivalence of graphs. Various families and results on chromatic uniqueness and chromatic equivalence of graphs have been obtained successively. The question on chromatic equivalence and uniqueness is called the *chromaticity* problem of graphs. Very recently, Dong, Koh and Teo finished a monograph on chromatic polynomials and chromaticity of graphs. So, this remains an active area of research. The reader can find more details on the chromaticity of graphs in the survey papers [17, 43, 44, 58, 68] and the monograph [23]. All chromatically unique graphs with 7 vertices and with 8 vertices can be found in [46].

In this thesis, the main aim is to study the algebraic properties of adjoint polynomials and the chromaticity of some classes of graphs. In Section 1.1, we introduce some basic definitions and terminology. In the rest of this introduction we describe our main results, together with some older results. Most of the terminology and notation used in this thesis is standard and can be found in the books [1] by Biggs and [6] by Bondy and Murty.

1.1 Basic definitions and knowledge

All *graphs* considered here are *finite* and *simple*. Let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$ denote the *vertex set*, the *edge set*, the number of vertices, the number of edges and the *complement* of a graph G, respectively. For a vertex v of a graph G, we denote by $N_G(v)$ the set of vertices of G which are adjacent to v and by $d(v)$ the degree of v. For an edge $e = v_1v_2$ of G, set $N_G(e) = N_G(v_1) \cup N_G(v_2) - \{v_1, v_2\}$ and $d(e) = d_G(e) = |N_G(e)|$. By $N_A(G)$ we denote the number of subgraphs isomorphic to K_3, i.e., the number of triangles in G. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies of H. $H \cong G$ means that H is isomorphic to G, denoted simply by $H = G$. We denote by $K_n - E(G)$ the graph obtained from the *complete graph* K_n by deleting all the edges of a graph isomorphic to G and by K_4^- the graph obtained from K_4 by deleting an edge. We denote the *path* and the *cycle* of n vertices respectively by P_n and C_n, and write $\mathcal{C} = \{C_n | n \geq 3\}, \mathcal{P} = \{P_n | n \geq 2\}$. By $K_{1,n-1}$ we denote the *star* of n vertices and $O_n = \overline{K_n}$.

For two polynomials $f(x)$ and $g(x)$ in x, by $(g(x), f(x))$ we denote the greatest common factor of $g(x)$ and $f(x)$, and by $g(x)|f(x)$ (respectively $g(x) \not| f(x)$) we denote that $g(x)$ divides $f(x)$ (respectively $g(x)$ does not divide

$f(x)$). Denote by $\partial f(x)$ the degree of $f(x)$. For a real number a, $\lfloor a \rfloor$ and $\lceil a \rceil$ denote respectively the largest integer smaller than or equal to a and the smallest integer larger than or equal to a.

For a graph G, a map $\phi : V \longrightarrow \{1, 2, \cdots, k\}$ is called a *vertex coloring* of G. A coloring is proper if no two adjacent vertices have the same color. A graph G is k-colorable if G has a proper k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G is k-colorable. The number of proper colorings of G with at most λ colors is called the *chromatic polynomial*, denoted by $P(G, \lambda)$.

For any graph G with p vertices, $P(G, \lambda)$ can be expressed in three ways:

$$P(G, \lambda) = \sum_{i=1}^{p} a_i(G)\lambda^i = \sum_{i=1}^{p} c_i(G)(\lambda)_{<i>} = \sum_{i=0}^{p} (-1)^{p-j} d_i(G)(\lambda)^{<i>},$$

where $(\lambda)_{<i>} = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - i + 1)$ and $(\lambda)^{<i>} = \lambda(\lambda + 1)(\lambda + 2)\cdots(\lambda + i - 1)$, for all $i \geq 1$. Then, the *σ-polynomial* of G is

$$\sigma(G, x) = \sum_{i=1}^{p} c_i(G)x^i$$

and the *τ-polynomial* of G is

$$\tau(G, x) = \sum_{i=1}^{p} d_i(G)x^i.$$

Remark 1.1.1. *The concept of σ–polynomial was first explicitly introduced and studied by Korfhage [45] in 1978. Actually, his definition of the σ–polynomial is equivalent to what we denote by $\sigma(G, x)/x^{\chi(G)}$. The reader may find the details on σ–polynomials in [30, 32, 49].*

An interpretation of the above coefficients a_i, c_i and d_i was given by Whitney [76], Read [65] and Brenti [4], respectively. Here, we recall the interpretation of the coefficients c_i as follows: For a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$ is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of $V(G)$. Then, $\alpha(G, r) = c_r$, that is, the chromatic polynomial of G is written as

$$P(G,\lambda) = \sum_{i\geq 1} \alpha(G,r)(\lambda)_{<i>}.$$

For a graph G with p vertices, if G_0 is a spanning subgraph of G and each component of G_0 is a complete graph, then G_0 is called *an ideal subgraph* of G. Let $b_i(G)$ denote the number of ideal subgraphs G_0 of G with $p-i$ components. It is clear that $b_0(G)=1$, $b_1(G)=q(G)$ and $b_i(G)=\alpha(\overline{G},p-i)$ for each i. In 1987, Liu introduced the *adjoint polynomial* of G as follows:

$$h(G,x) = \sum_{i=0}^{p-1} b_i(G)x^{p-i}.$$

From the above argument, we get

$$\sigma(\overline{G},x) = h(G,x) \tag{1.1}$$

and

$$P(\overline{G},\lambda) = \sum_{i=0}^{p-1} b_i(G)(\lambda)_{<p-i>}. \tag{1.2}$$

Remark 1.1.2. *Using the term special graphs, Frucht gave the expression (1.2) in [38]. The adjoint polynomial of a graph is a special subgraph polynomial, see [35].*

Example 1.1.1. *Let* $\overline{G} = P_3 \cup P_2$*, Then* $h(\overline{G},x) = x^5 + 3x^4 + 2x^3$*. So, we have*

$$P(G,\lambda) = \lambda_{<5>} + 3(\lambda)_{<4>} + 2\lambda_{<3>}.$$

The properties of the chromatic polynomial of G and its σ-polynomial (or the adjoint polynomial of $\overline{G}$) have a close relation with their roots, which was studied by Brenti, Royle and Wagner in 1992 and 1994. In Chapters 2 and 3 of this thesis, we investigate some algebraic properties of the adjoint polynomials of some graphs, such as recursive relations, divisibility, roots of the adjoint polynomials and so on.

Two graphs G and H are said to be *chromatically equivalent* (or simply χ-*equivalent*), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. It is clear that "$\sim$" is an equivalence relation on the family of all graphs. By $[G]$ we denote the equivalence class determined by G under "$\sim$". A graph G is called *chromatically unique* (or simply χ-*unique*) if $H \cong G$ whenever $H \sim G$. For a set $\mathcal{G}$ of graphs, if $[G] \subset \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is called χ-*closed*.

The problem of chromaticity of a graph G is to consider the following:
(i) Is G χ-unique?
(ii) Determine $[G]$ if G is not χ-unique.

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim_h H$, if $h(G,x) = h(H,x)$. Clearly, "$\sim_h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H | H \sim_h G\}$. A graph G is said to be *adjointly unique* if $H \cong G$ whenever $H \sim_h G$. For a set $\mathcal{G}$ of graphs, if $[G]_h \subset \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is called *adjointly closed*.

From the above definitions, we have

Theorem 1.1.1. *(i)* $G \sim H$ *if and only if* $\overline{G} \sim_h \overline{H}$;
(ii) $[G] = \{H | \overline{H} \in [\overline{G}]_h\}$;
(iii) G *is adjointly unique if and only if* $\overline{G}$ *is* χ-*unique.*

So, from Theorem 1.1.1, one sees that the study of chromaticity of a graph $\overline{G}$ is equivalent to investigate the following problems:
(i) Is G adjointly unique?
(ii) Determine $[G]_h$ if G is not adjointly unique.

In Chapters 4 to 6, we investigate the chromaticity of some classes of graphs, which are some dense graphs, complete multipartite graphs and general multipartite graphs. Some new results are obtained and some open problems and conjectures are solved.

At the end of this section, we point out that all parameters used throughout the thesis take nonnegative integer values.

1.2 The adjoint polynomials of graphs

For convenience, we denote $h(G,x)$ by $h(G)$ for a graph G. In 1987, Liu introduced the definition of adjoint polynomial of a graph and gave two important properties in the following theorems.

Theorem 1.2.1. *([50]) Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i).$$

For an edge $e = v_1v_2$ of a graph G, the graph $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G)\backslash\{v_1, v_2\}) \cup \{v\}$, and the edge set of $G * e$ is $\{e'|e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv|u \in N_G(v_1) \cap N_G(v_2)\}$. For example, let e_1 be an edge of C_4 and e_2 an edge of K_4, then $C_4 * e_1 = P_2 \cup K_1$ and $K_4 * e_2 = K_3$.

Theorem 1.2.2. *([51]) Let G be a graph with $e \in E(G)$. Then*

$$h(G,x) = h(G-e,x) + h(G*e,x).$$

In particular, if $e = u_1u_2$ does not belong to any triangle of G, then

$$h(G,x) = h(G-e,x) + xh(G - \{u_1,u_2\}, x),$$

where $G - e$ (respectively $G - \{u_1, u_2\}$) denotes the graph obtained by deleting the edge e (respectively the vertices u_1 and u_2) from G.

Let D_n, F_n, U_n, B_n, A_n be the graphs with n vertices, shown in Figure 1.1.

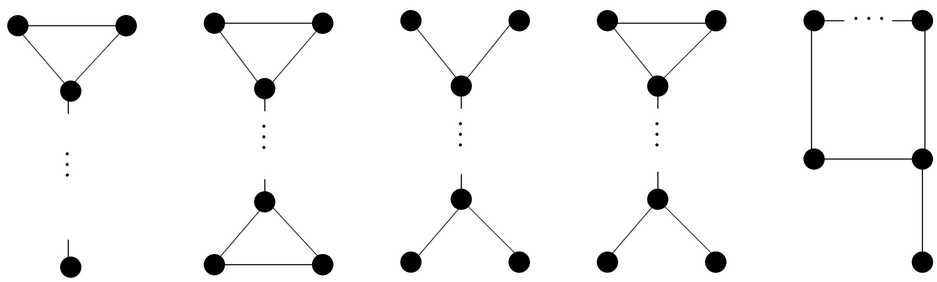


Figure 1.1

By applying the above theorems, Liu and Zhao gave the following results.

Theorem 1.2.3. *([58, 62])) (i) For all* $n \geq 6$, $h(C_n) = x(h(C_{n-1})+h(C_{n-2}))$;
(ii) For all $n \geq 3$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$;
(iii) For all $n \geq 6$, $h(D_n) = x(h(D_{n-1}) + h(D_{n-2}))$;
(iv) For all $n \geq 8$, $h(F_n) = x(h(F_{n-1}) + h(F_{n-2}))$.

Theorem 1.2.4. *([58]) (i) For all* $n \geq 2$, $h(P_n) = \sum\limits_{k \leq n} \binom{k}{n-k} x^k$;
(ii) For all $n \geq 4$, $h(C_n) = \sum\limits_{k \leq n} \frac{n}{k} \binom{k}{n-k} x^k$;
(iii) For all $n \geq 4$, $h(D_n) = \sum\limits_{k \leq n} \left(\frac{n}{k} \binom{k}{n-k} + \binom{k-2}{n-k-3} \right) x^k$.

Theorem 1.2.5. *([56, 58]) (i) For* $n \geq 3$, $h(P_{2n+1}) = h(P_n)h(C_{n+1})$;
(ii) For $n \geq 4$, $h(C_n \cup K_1) = h(T_{1,1,n-2})$;
(iii) For $n \geq 4$, $h(D_n \cup K_1) = h(T_{1,2,n-3})$.

Let G be a graph. For $e \in E(G)$, by $G_e(P_m)$ we denote the graph obtained from G by replacing the edge e by P_m. In Section 2.2, we give the following general result.

Theorem 1.2.6. *If* $m \geq 4$, *then*

$$h(G_e(P_m)) = x\left(h(G_e(P_{m-1})) + h(G_e(P_{m-2}))\right).$$

The divisibility of the adjoint polynomials is very important for studying the chromaticity of dense graphs, as will become clear in Chapter 4. Liu and Wang obtained the following results.

Theorem 1.2.7. *(i) ([54, 58]) For $n, m \geq 2$, $h(P_n)|h(P_m)$ if and only if $(n+1)|(m+1)$;*
(ii) ([75]) For $n \geq 2$ and $m \geq 4$, $h(P_n)|h(D_m)$ if and only if $n = 2$ and $m \equiv 2(mod\ 3)$, or $n = 4$ and $m \equiv 3(mod\ 5)$.

In Section 2.2, we give some more general results.

Theorem 1.2.8. *Let $\{g_i(x)\}_i$ $(i \geq 0)$ be a sequence of polynomials with integral coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*
(i) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$;
(ii) $h_1(P_n)|g_{n+1+i}(x)$ if and only if $h_1(P_n)|g_i(x)$, for any positive integers n and i.

By Theorems 1.2.6 and 1.2.8, it is not hard to obtain a sufficient and necessary condition of the form $h(P_n)|h(H)$ for any $H \in \{P_n, C_n, D_n, F_n, U_n, A_n, B_n\}$. For example, $h(P_n)|h(A_m)$ if and only if $n = 2$ and $m \equiv 2(mod\ 3)$, for $n \geq 2$ and $m \geq 6$.

Let $T_{a,b,c}$ denote a tree with a vertex v of degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$, shown in Figure 1.3. By Theorems 1.2.6 and 1.2.8, in Section 2.2 we get the following results.

Theorem 1.2.9. *For $k \geq 1$ and $t \geq 1$ such that $kt > 3$, we have that $h(P_{t-1})|h(T_{1,t,kt-3})$, $h(P_t)|h(T_{1,t,kt+k-1})$ and $h(P_{t+2})|h(T_{1,t,k(t+3)})$.*

Theorem 1.2.10. *For $l \geq 2$, $m \geq 1$ and $k \geq 1$, we have:*
(i) $h(P_l)|h(T_{1,1,m})$ if and only if $(l, m) \in \{(3, 4k)\}$;
(ii) $h(P_l)|h(T_{1,2,m})$ if and only if $(l, m) \in \{(2, 3k-1), (4, 5k)\}$;
(iii) $h(P_l)|h(T_{1,3,m})$ if and only if $(l, m) \in \{(2, 3k), (3, 4k-1), (5, 6k)\}$;
(iv) $h(P_l)|h(T_{1,4,m})$ if and only if $(l, m) \in \{(3, 4k-3), (4, 5k-1), (6, 7k)\}$;
(v) $h(P_l)|h(T_{1,5,m})$ if and only if $(l, m) \in \{(2, 3k-1), (3, 4k), (4, 5k-3), (5, 6k-1), (7, 8k)\}$;
(vi) $h(P_l)|h(T_{1,6,m})$ if and only if $(l, m) \in \{(2, 3k), (5, 6k-3), (6, 7k-1), (8, 9k)\}$.

For a graph G, if every $\chi(G)$-coloring of G gives the same partition of $V(G)$, then G is said to be *a uniquely $\chi(G)$–colorable graph*. Chao and Chen [12, 13], Chao [14] and Chia [16] found some uniquely n-colorable graphs. As an application of the recursive relations of adjoint polynomials, in Section 2.3 we generalize all of Chao's results in [14] by the following results.

Theorem 1.2.11. *Let s be an odd integer. Then $\overline{G(K_m, P_s)}$ is a uniquely $\frac{s+1}{2}$–colorable graph with $m + s - 1$ vertices, where $G(K_m, P_s)$ denotes the graph obtained by identifying a vertex of K_m with an end-vertex of P_s.*

Corollary 1.2.1. *([14]) For any $n \geq 1$ and $m \geq 2$, we have*
(i) $\overline{P_{2n}}$ is a uniquely n-colorable graph with $2n$ vertices;
(ii) $\overline{D_{2m+1}}$ is a uniquely m-colorable graph with $2m + 1$ vertices.

Theorem 1.2.12. *Let s be an odd integer. Then $\overline{G(K_m, P_s) \cup K_1}$ and $\overline{G'(K_m, P_s)}$ are uniquely $\frac{s+3}{2}$-colorable graphs that are chromatically equivalent, where $G'(K_m,P_s)$ denotes a vertex splitting graph obtained from $G(K_m, P_s)$ (see Definition 2.3.1).*

Corollary 1.2.2. *Let $n \geq 3$. There exist infinitely many uniquely n-colorable graphs that are chromatically equivalent.*

We now turn our attention to the roots of adjoint polynomials. In 1992, Brenti investigated the roots of chromatic polynomials and of five other related polynomials. Here we present several results on roots of an adjoint polynomial.

For a graph G with vertex set $V(G)$ and edge set $E(G)$, G is called a *comparability graph* if there exists a partial order $\preceq$ on $V(G)$ such that $uv \in E(G)$ if and only if $u \neq v$ and either $u \preceq v$ or $v \preceq u$, see [39].

Theorem 1.2.13. *([4, 5])*
(i) Let G be a comparability graph. Then all roots of $h(G, x)$ are real;
(ii) Let G be a graph without triangles. Then all roots of $h(G, x)$ are real;
(iii) All roots of $h(K_n, x)$ are real for any integer n.

For a graph G, by $\beta(G)$ we denote the minimum real roots of $h(G, x)$. In Section 3.2, we give some fundamental equalities and inequalities on the minimum real roots of $h(G, x)$. We determine all connected graphs G such that $\beta(G) \in (-(2 + \sqrt{5}), -4) \cup [-4, 0]$ in Sections 3.3 and 3.4.

Let $\mathcal{T}_1 = \{T_{1,1,n} | n \geq 1\}$ and $\mathcal{U} = \{U_n | n \geq 6\}$. In Figures 1.2 and 1.3 we list some graphs used in rest of this thesis.

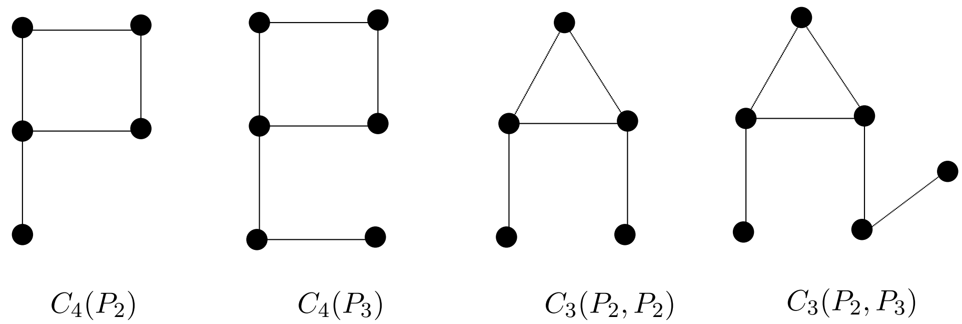


Figure 1.2

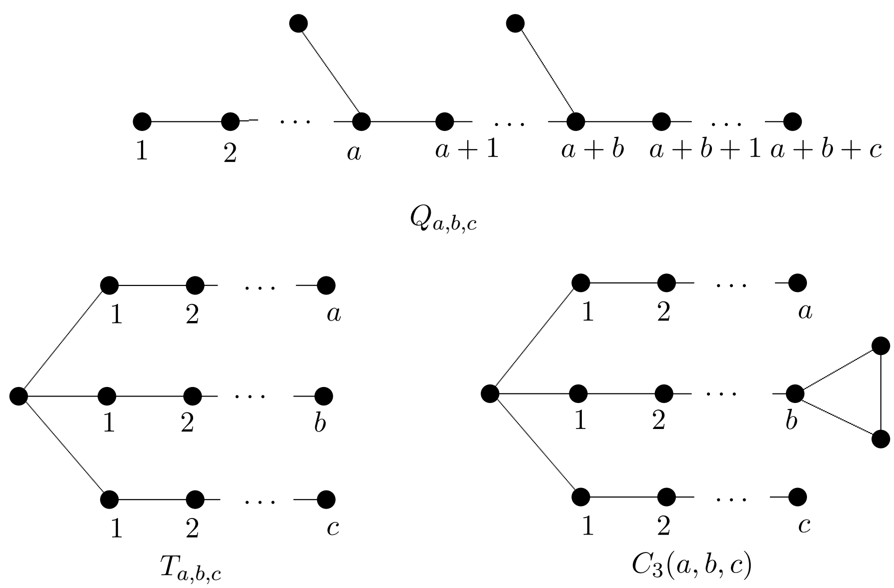


Figure 1.3

The following theorems can be found in Sections 3.2 to 3.4.

Theorem 1.2.14. *Let G be a connected graph and let H be a proper subgraph of G. Then*

$$\beta(G) < \beta(H).$$

Theorem 1.2.15. *Let G be a connected graph. Then*
(i) $\beta(G) = -4$ if and only if

$$G \in \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2, P_2), K_4^-, D_8\} \cup \mathcal{U};$$

(ii) $\beta(G) > -4$ *if and only if*

$$G \in \{K_1\} \cup \{T_{1,2,i} | i = 2,3,4\} \cup \{D_i | i = 4,5,6,7\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1.$$

Theorem 1.2.16. *Let G be a connected graph. Then $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the following graphs:*
(i) $T_{a,b,c}$ for $a = 1, b = 2, c > 5$, or $a = 1, b > 2, c > 3$, or $a = b = 2, c > 2$, or $a = 2, b = c = 3$;
(ii) $Q_{a,b,c}$ for $(a,b,c) \in \{(2,1,2),(3,4,2),(3,5,3),(4,7,3),(4,8,4)\}$, or $a \geq 2, b \geq b^*(a,c), c \geq 1$, where $(a,c) \neq (2,1)$ and

$$b^*(a,c) = \begin{cases} a+c+1 & for \quad a \geq 4, \\ 3+c & for \quad a = 3, \\ c & for \quad a = 2; \end{cases}$$

(iii) D_n for $n \geq 9$;
(iv) A_n for $n \geq 6$;
(v) F_n for $n \geq 9$;
(vi) $C_3(a,b,c)$ for $a = 1, b = 5$ and $c = 3$, or $a = 1$ and $b \geq 1$ if $c = 1$, or $a = 1$ and $b \geq 4$ if $c = 2$, or $a = 1$ and $b \geq c+3$ if $c \geq 3$;
(vii) $G \cong C_4(P_3)$, or $G \cong C_3(P_2, P_3)$.

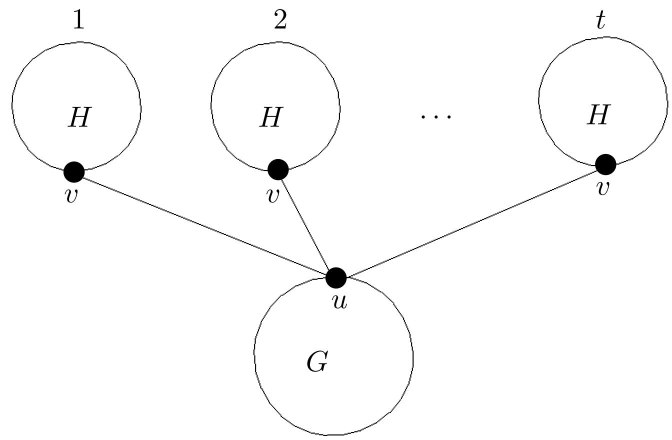
As byproduct of the above theorems, the following corollary is obtained.

Corollary 1.2.3. *Let G be a connected graph with $\beta(G) \geq -(2+\sqrt{5})$. Then all the roots of $\sigma(\overline{G}, x)$ are real.*

A graph G is called *P-real (or σ-real)* if all roots of $P(G,x)$ (or $\sigma(G,x)$) are real; otherwise G is called *P-unreal (or σ-unreal)*. For a connected graph G with n vertices, we define $\eta(G) = |E(G)|/\binom{n}{2}$, where $\eta(G)$ is said to be *the edge-density* of G. We denote by $\eta(n)$ the minimum edge-density over all n vertices graphs with σ-unreal roots. Brenti, Royle and Wagner in 1994 determined all σ-unreal graphs with 8 and 9 vertices. Furthermore, they proposed the following problem.

Problem 1.2.1. *([5]) For a positive integer n, let $\eta(n)$ be the minimum edge-density over all σ-unreal graphs with n vertices. Give a good lower bound for $\eta(n)$. In particular, is there a constant $c > 0$ such that $\eta(n) > c$ for sufficiency large n?*

Let H and G be two graphs and let $v \in V(H)$ and $u \in V(G)$. We denote by $G_u^t(H_v)$ the graph obtained from G and t copies of H and a star $K_{1,t}$ by identifying every vertex of degree 1 of $K_{1,t}$ with vertex v of a copy of H and identifying the center of $K_{1,t}$ with vertex u of G, as shown in Figure 1.4.



$G_u^t(H_v)$

Figure 1.4

For two positive integers n and s, we denote by $K_n - s$ the graph obtained by deleting s edges from K_n. In Section 3.5, we establish a way of constructing σ-unreal graphs and give a negative answer to Problem 1.2.1 by the following theorems.

Theorem 1.2.17. *Let H be a graph with m vertices and $v \in V(H)$ such that $\overline{H}$ is σ-unreal. Let t be a positive integer and $H_i = K_{n-mi}$. Then there exists a σ-unreal graph sequence $\overline{H_1 \cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}$, $\ldots$, $\overline{H_t^t(H_v)}$ such that $\eta(\overline{H_1 \cup H}) \to 0$ and $\eta(\overline{H_i^i(H_v)}) \to 0$ as $n \to \infty$, where $i = 2, 3, \ldots, t$, moreover $\eta(n) \to 0$ as $n \to \infty$.*

Theorem 1.2.18. *Let H be a graph with m vertices and $v \in V(H)$ such that $\overline{H}$ is σ-unreal. Let t be a positive integer and $H_i = K_{n-mi} - \frac{p}{2q}[(n-im)^2 - s_i]$, where $i = 1, 2, \ldots, t$ and $(n-im)^2 \equiv s_i (mod\, 2q)$. Then for any rational number p/q, $0 \le p/q \le 1$, there exists a σ-unreal graph sequence $\overline{H_1 \cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}$, $\ldots$, $\overline{H_t^t(H_v)}$ such that $\eta(\overline{H_1 \cup H}) \to p/q$ and $\eta(\overline{H_i^i(H_v)}) \to p/q$ as $n \to \infty$, where $i = 2, 3, \ldots, t$.*

1.3 The chromaticity of some dense graphs

Let G be a graph and let $h(G,x) = x^{\ell(G)} h_1(G,x)$ such that $h_1(G,x)$ is a polynomial with a nonzero constant term, where $\ell(G)$ is the degree of the lowest term of $h(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called an *irreducible graph*.

In 1994, Liu [55] found a necessary and sufficient condition for C_n to be irreducible and gave some irreducible graphs, see [58]. Later, Zhao and Liu [82] obtained a necessary and sufficient condition for P_n to be irreducible. By applying the unique factorization theorem of polynomials over the rational number field and using the fundamental properties and irreducibility of adjoint polynomials, Li, Liu, Wang, Ye, Zhao *et al.* found many new families of chromatically unique graphs. We summarize their results as follows:

Theorem 1.3.1. *([52, 60, 61]) Let $G = \bigcup_{i=1}^{r} k_i P_{n_i} \cup \bigcup_{j=1}^{s} l_j C_{m_j}$, where $n_i \geq 4$ and $m_j \geq 5$. If P_{n_i} and C_{m_j} are irreducible for each i, $1 \leq i \leq r$, and for each j, $1 \leq j \leq s$, then $K_n - E(G)$ is χ-unique, where $n \geq V(G)$.*

Theorem 1.3.2. *([54]) If q, $q > 5$, is a prime number, then $\overline{P_{q-1}}$ is χ-unique.*

Conjecture 1.3.1. *([58]) For every even number $n, n > 4$, $\overline{P_n}$ is χ-unique.*

The above conjecture was confirmed respectively by Zhao, Huo and Liu [81] and by F.M. Dong, K.L. Teo, C.H.C. Little and M.D. Hendy [29]. By applying the theory of matching polynomials, see [34, 36, 37], Liu and Bao in 1993 obtained the following result.

Theorem 1.3.3. *([59]) If G is a 2-regular graph without C_3 and C_4, then $\overline{G}$ is χ-unique.*

In 1996, Du improved the above results and got the following results.

Theorem 1.3.4. *([32]) If $n_i \not\equiv 4(mod\ 10)$ and n_i is even, $m_j \geq 3$ and $m_j \neq 4$, then $\overline{lK_3 \cup (\cup_i P_{n_i})}$ and $\overline{\cup_j C_{m_j}}$ are χ-unique, where $l \geq 0$.*

Very recently, Dong, Teo, Little and Hendy [29] investigated the chromaticity of complements of $H = aK_3 \cup bD_4 \cup \bigcup\limits_{1 \leq i \leq s} P_{u_i} \cup \bigcup\limits_{1 \leq j \leq t} C_{v_j}$, where $a, b \geq 0, u_i \geq 3, u_i \not\equiv 4(mod\ 5), v_j \geq 4$ and obtained a necessary and sufficient condition for $\overline{H}$ to be chromatically unique.

Theorem 1.3.5. *([29]) For nonnegative integers $r_0, r_1, s, a_1, a_2, \cdots, a_s$, the graph* $\overline{r_0K_1 \bigcup r_1K_3 \bigcup_{i=1}^{s} a_iP_{2i}}$ *is χ-unique if and only if $r_0r_1 + a_2 = 0$.*

Theorem 1.3.6. *([29]) Let $G = aK_3 \bigcup bD_4 \bigcup\limits_{1\le i\le s} P_{u_i} \bigcup\limits_{1\le j\le s} C_{v_j}$ for nonnegative integers a, b, u_i and v_j with $u_i \ge 3$, $u_i \not\equiv 4(mod\ 5)$ and $v_j \ge 4$. Then $\overline{G}$ is χ-unique if and only if $b = 0$, $v_j \ge 5$, u_i is even when $u_i \ge 6$ and $\{u_1+1, u_2+1, \cdots, u_s+1\} \bigcap \{v_1, \cdots, v_t\} = \phi$.*

However, a necessary and sufficient condition for $\overline{aK_1 \cup_i P_i \cup_j C_j}$ to be chromatically unique is not known. In Section 4.3, we give a necessary and sufficient condition for $\overline{aK_1 \cup_i P_i \cup_j C_j}$ and $\overline{\cup_i U_{n_i}}$ to be χ-unique by the following theorems.

Theorem 1.3.7. *Let $A = \{n | n \equiv 0(mod\ 2)$ and $n \ge 6\}$ and $B = \{n | n \ge 5\}$. For a graph G with p vertices and $\delta(G) \ge p-3$, we have that G is χ-unique if and only if $\overline{G}$ is one of the following graphs:*

(i) $rK_1 \cup (\bigcup\limits_{1\le i\le s} P_{n_i})$ for $r = 0$ and $n_i \in A \cup \{2,3,5\}$, or $r \ge 1$ and $n_i \in A \cup \{2,3\}$, where $r, s \ge 0$;

(ii) $t_1P_2 \cup (\bigcup\limits_{1\le i\le s} P_{n_i}) \cup (\bigcup\limits_{1\le j\le t} C_{m_j}) \cup lC_3$ for $t_1 = 0$ and $M = \phi$, or $t_1 \ge 1$ and $(\{6,9,15\} \cap B) \cup M = \phi$, where $t_1, l, s, t \ge 0$, $t + l \ge 1$, $n_i \in A \cup \{3,5\}$, $m_j \in B$ and $M = (A \cup \{3,5\}) \cap \{n-1 | n \in B\}$.

Theorem 1.3.8. *Let $n_i \ge 6$. Then $\overline{\cup_{i=1}^{m} U_{n_i}}$ is χ-unique if and only if $n_i = 7$ or $n_i \ge 10$ and n_i is even, where $i = 1, 2, \cdots, m$.*

Dong, Teo, Little and Hendy [29] also determined all adjointly equivalent classes of graphs $r_0K_1 \cup r_1K_3 \cup \bigcup\limits_{1\le i\le s} P_{2l_i}$, for $r_0, r_1 \ge 0, l_i \ge 1$, and obtained a necessary and sufficient condition for two graphs H and G in $\mathcal{G}_1$ to be adjointly equivalent, where

$$\mathcal{G}_1 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\le i\le s} P_{u_i} \right.$$

$$\left. \cup \bigcup_{1\le j\le t} C_{v_j} | a, b, s, t \ge 0, u_i \ge 3, u_i \not\equiv 4(mod\ 5), v_j \ge 4 \right\}.$$

Let

$$\mathcal{G}_2 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\le i\le s} P_{u_i} \cup \bigcup_{1\le j\le t} C_{v_j} | a,b,s,t \ge 0, u_i \ge 3, v_j \ge 4 \right\}$$

and

$$\mathcal{G}_3 = \left\{ rK_1 \cup \bigcup_{1\le j\le t} C_{v_j} | r,t \ge 0, v_j \ge 4 \right\}.$$

Indeed, it is not easy to determine the equivalence class of each graph in $\mathcal{G}_i$ for $i = 1,2,3$. So, they proposed the following problem: For a set $\mathcal{G}$ of graphs, determine

$$min_h\mathcal{G} = \bigcup_{G\in\mathcal{G}} [G]_h,$$

where $min_h\mathcal{G}$ is called *the adjoint closure* of $\mathcal{G}$. They also proposed the following problem and conjectures.

Problem 1.3.1. *([29]) Determine $min_h(\mathcal{G}_2)$.*

Conjecture 1.3.2. *([29]) The following set equality holds.*

$$min_h(\mathcal{G}_2) \equiv \left\{ rK_1 \cup aK_3 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le i\le s} P_{u_i} \right.$$

$$\left. \cup \bigcup_{1\le j\le t} C_{v_j} | a,b,r,s,t \ge 0, m+r \le a, r_i \ge 2, u_i \ge 3, v_j \ge 4 \right\}.$$

Conjecture 1.3.3. *([29]) The following set equality holds.*

$$min_h(\mathcal{G}_3) \equiv \left\{ rK_1 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le j\le t} C_{v_j} | r,b,m,t \ge 0, r_i \ge 2, v_j \ge 4 \right\}.$$

Let

$$\mathcal{F}_1 = \{K_1\} \cup \{T_{1,2,i}, D_{i+3} | i = 1,2,3,4\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1$$

and

$$\mathcal{F}_2 = \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8\} \cup \mathcal{U}.$$

Take $\mathcal{Y}_1 = \{\bigcup G_i | G_i \in \mathcal{F}_1\}$ and $\mathcal{Y}_2 = \{\bigcup G_i | G_i \in \mathcal{F}_1 \cup \mathcal{F}_2\}$.

In Section 4.4, we obtain a method for determining the adjoint equivalence class of each graph in sets $\mathcal{Y}_i$, where $i = 1, 2$. Furthermore, we give a negative answer to Conjectures 1.3.2 and 1.3.3 by the following theorem.

Theorem 1.3.9. *(i) There exists a graph G in $min_h\mathcal{G}_2$ such that G contains each graph in $\mathcal{F}_1$ as its component and $min_h\mathcal{G}_2 \subseteq \mathcal{Y}_1$.*
(ii) Let $\mathcal{F}_3 = \{K_1\} \cup \{T_{1,2,i}, D_{i+3} | i = 1, 3, 4\} \cup \{C_i | i \geq 4\} \cup \mathcal{T}_1$ and $\mathcal{Y}_3 = \{\bigcup G_i | G_i \in \mathcal{F}_3\}$. Then there exists a graph G in $min_h\mathcal{G}_3$ such that G contains each graph in $\mathcal{F}_3$ as its component and $min_h\mathcal{G}_3 \subseteq \mathcal{Y}_3$.

In order to study adjoint uniqueness of complex graphs, Liu and Zhao [58, 62] and Du [31, 32] introduced and studied independently an invariant on adjointly equivalent graphs as follows.

For a graph G with q edges, its *character* $R_1(G)$ or *invariant* $R_1(G)$ is defined as

$$R_1(G) = \begin{cases} 0, & \text{if } q = 0, \\ b_2(G) - \binom{b_1(G)-1}{2} + 1, & \text{if } q > 0, \end{cases}$$

where $b_1(G)$ and $b_2(G)$ are the second and the third coefficients of $h(G)$, respectively.

In 1996, Liu and Zhao investigated the adjoint uniqueness of graphs $\bigcup_i C_i \cup \bigcup_j D_j \cup \bigcup_k F_k$ and of graphs $\bigcup_{k \leq s \leq t} T_{k,s,r}$. They obtained the following results.

Theorem 1.3.10. *([57]) Let $G = \bigcup_{n_i} D_{n_i}$, where $n_i \geq 5$ is a positive integer. If D_{n_i} is an irreducible graph for all i, then $\overline{G}$ is χ-unique.*

Theorem 1.3.11. *([62]) Let $n \geq 6$. If F_n is an irreducible graph, then $\overline{F_n}$ is χ-unique.*

In [58], Liu proposed the following problems.

Problem 1.3.2. *([58]) Study the adjoint uniqueness of $\bigcup_i P_{n_i} \cup \bigcup_j C_{m_j} \cup \bigcup_k D_{t_k}$.*

Problem 1.3.3. *([58]) Study the adjoint uniqueness of graphs G with $R_1(G) = -1$.*

Li, Bao and Liu [47], Liu [56], Wang [73] and Wang and Liu [74] obtained the following results.

Theorem 1.3.12. *([73]) Let* $G = \bigcup_i P_{n_i} \cup \bigcup_{2\le k\le s\le t} T_{k,s,t}$*, where* t, s, k*, and* n_i *are positive integers and* $n_i \ge 2$ *for each* i*. If all* P_{n_i} *and* $T_{k,s,k}$ *are irreducible, then* $\overline{G}$ *is* χ*-unique.*

Theorem 1.3.13. *([74]) Let* $G = \bigcup_{1,l_1,l_2} T_{1,l_1,l_2} \cup \bigcup_{k\le s\le t} T_{k,s,t}$*, where* $t \ge s \ge k \ge 2$ *and* $l_2 \ge l_1 \ge 3$*. If all* $T_{k,s,t}$ *and* T_{1,l_1,l_2} *are irreducible, then* $\overline{G}$ *is* χ*-unique.*

In Sections 4.5 to 4.7, we focus on studying adjoint uniqueness of some complex graphs. Moreover some new results on Problems 1.3.2 and 1.3.3 are obtained. First, in Section 4.5, we study adjoint uniqueness of $(\cup_i C_{n_i}) \cup (\cup_i D_{m_j}) \cup (\cup_{a,b} T_{1,a,b})$ and of $rK_1 \cup (\bigcup_{a,b} T_{1,a,b})$. We give many new chromatically unique graphs as follows.

Theorem 1.3.14. *Let* $n_i \ge 5$ *and* $m_j \ge 9$*, for each* i *and* j*, and let* $3 \le l_1 \le 10$ *and* $l_1 \le l_2$*. Let* $G = (\cup_i C_{n_i}) \cup (\cup_j D_{m_j}) \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$*. If* $h(P_n) \nmid h(C_{n_i})$*,* $h(P_n) \nmid h(D_{m_j})$ *and* $h(P_n) \nmid h(T_{1,l_1,l_2})$*, for all* $n \ge 2$*, then* $\overline{G}$ *is* χ*–unique if and only if* $l_2 \ne 2l_1 + 5$ *and* $(l_1, l_2) \ne (n_i - 1, n_i)$*, for all* i*.*

Corollary 1.3.1. *Let* $G_i \in \{C_i | i \ge 5, i \not\equiv 2(mod\ 4)\} \cup \{D_j | j \ge 9, j \not\equiv 2(mod\ 3), j \not\equiv 3(mod\ 5)\} \cup \{T_{1,l_1,l_2} | 3 \le l_1 \le 6, l_1 \le l_2, l_1 \ne l_2, l_1 \ne l_2 + 1, l_2 \ne 2l_1 + 5\}$ *and* $(l_1, l_2) \notin \{(3, 3k), (3, 4k-1), (4, 4k+1), (4, 5k-1), (4, 7k), (5, 3k+2), (5, 4k+4), (5, 5k+2), (6, 3k+3), (6, 7k-1) | k \ge 1\}$*. Then* $\overline{\cup_i G_i}$ *is* χ*-unique.*

Theorem 1.3.15. *Let* $3 \le l_1 \le 10$ *and* $l_1 \le l_2$*. If* $h(P_m) \nmid h(T_{1,l_1,l_2})$ *for any* $m \ge 2$*, then* $K_n - E(\cup_{l_1,l_2} T_{1,l_1,l_2})$ *is* χ*-unique if and only if* $l_2 \ne 2l_1 + 5$*, where* $n \ge \sum_{l_1,l_2} |V(T_{1,l_1,l_2})|$*.*

Corollary 1.3.2. *Let* $G_i \in \{T_{1,l_1,l_2} | 3 \le l_1 \le 6, l_1 \le l_2, l_1 \ne l_2, l_1 \ne l_2 + 1, l_2 \ne 2l_1 + 5\}$ *and* $(l_1, l_2) \notin \{(3, 3k), (3, 4k-1), (4, 4k+1), (4, 5k-1), (4, 7k), (5, 3k+2), (5, 4k+4), (5, 5k+2), (6, 3k+3), (6, 7k-1) | k \ge 1\}$*. Then* $K_n - E(\cup_i G_i)$ *is* χ*-unique, where* $n \ge \sum_{l_1,l_2} |V(T_{1,l_1,l_2})|$*.*

In Section 4.6, we obtain the following results.

Theorem 1.3.16. *Let $G = A_n \cup (\cup_{i=1}^{k} C_{m_i})$, where $m_i \not\equiv 2(mod\ 4)$ and $m_i \geq 5$ for all i, and $n \not\equiv 2(mod\ 3)$ and $n \geq 5$. Then $[G]_h = \{G\}$ except for $[G]_h = \{A_7 \cup (\cup_{i=1}^{k} C_{m_i}), B_7 \cup (\cup_{i=1}^{k} C_{m_i})\}$ for $n = 7$; in particular, $\overline{G}$ is χ-unique if and only if $n \neq 7$.*

Theorem 1.3.17. *Let $G = B_n \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})$, where $m_i \not\equiv 2(mod\ 4)$, and $m_i \geq 5$ and $m_i \neq 9, 15$, for all i and $n \geq 7$. Then $[G]_h = \{G\}$ except for the following cases:*
(i) $[G]_h = \{G, A_7 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 7$;
(ii) $[G]_h = \{G, F_{13} \cup T_{1,1,1} \cup (a-1)C_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 8$ and $a \geq 1$;
(iii) $[G]_h = \{G, F_{15} \cup T_{1,1,1} \cup C_5 \cup aC_9 \cup (b-1)C_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 9$ and $b \geq 1$;
(iv) $[G]_h = \{G, A_6 \cup C_4 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i}), A_6 \cup D_4 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 10$.
In particular, $\overline{G}$ is χ-unique if and only if $n \neq 7, 10$, and $a = 0$ when $n = 8$, and $b = 0$ when $n = 9$.

In Section 4.7 we introduce a new invariant $R_3(G)$, that is $R_3(G) = R_1(G) + q(G) - p(G)$, and give some properties. We prove the following results in that section.

Theorem 1.3.18. *Let G and H be two graphs such that $h(G,x) = h(H,x)$. Then*

$$R_3(G) = R_3(H).$$

Theorem 1.3.19. *Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then*

$$R_3(G) = \sum_{i=1}^{k} R_3(G_i).$$

Theorem 1.3.20. *Let G be a connected graph. Then*
(i) $R_3(G) \leq 1$, and the equality holds if and only if $G \cong K_3$;
(ii) $R_3(G) = 0$ if and only if $G \in \mathcal{L} \backslash \{K_3\}$, where $\mathcal{L} = \{K_3, K_4^-, K_4, P_n, C_{n+2}, D_{n+2}, F_{n+4} | n \geq 2\}$.

Theorem 1.3.21. *Let $\mathcal{F}_a = \{aK_3 \cup \bigcup_i G_i | G_i \in \mathcal{L}$ and $h(K_3) \nmid h(G_i)\}$, where a is a nonnegative integer. Then $\mathcal{F}_a$ is adjointly closed.*

Theorem 1.3.22. *Let a, t, r be nonnegative integers and let $G = (\cup_{i \in A} P_i) \cup (\cup_{j \in B} C_j) \cup (\cup_{k \in M} D_k) \cup (\cup_{s \in E} F_s) \cup aK_3 \cup tK_4^- \cup rK_4$, where $i \geq 2$, $i \not\equiv 4(mod\ 5)$ and i is even, $j \geq 5$, $k \not\equiv 3(mod\ 5)$ and $k \geq 9, s \not\equiv 2(mod\ 5)$ and $s \geq 6$. Then $\overline{G}$ is χ–unique if and only if $j \neq i+1$ if $2 \notin A$, or $j \neq 6, 9, 15$ and $j \neq i+1$ if $2 \in A$.*

1.4 The chromaticity of multipartite graphs

It is well known that all complete graphs K_n are χ-unique, for $n \geq 1$. A natural question is: which complete t-partite graphs $K(n_1, n_2, \cdots, n_t)$, for $n_i \geq 1$, are χ-unique? The following result was obtained by Chao and Novacky in 1982.

Theorem 1.4.1. *([7]) For $t \geq 2$, the complete t-partite graph $K(n_1, n_2, \cdots, n_t)$ is χ-unique if $|n_i - n_j| \leq 1$, for all $i, j = 1, 2, \cdots, t$.*

In 1978, Chao gave the following conjecture.

Conjecture 1.4.1. *For $n \geq 2$ and $0 \leq k \leq 2$, the graph $K(n, n+k)$ is χ-unique.*

This was later confirmed by Salzberg, López and Giudici in 1986. In fact, they proved more general results.

Theorem 1.4.2. *([69]) The graph $K(n, n+k)$ is χ-unique, for all $n \geq 2$ and $0 \leq k \leq max\{5, \sqrt{2}n\}$.*

Conjecture 1.4.2. *([69]) All complete bipartite graphs $K(n, m)$ are χ-unique when $n \geq m \geq 2$.*

The conjecture was finally resolved completely by Teo and Koh in 1990 [71]. A couple of alternative proofs were given by Dong [22] and Teo and Koh [72].

Theorem 1.4.3. *([71]) All complete bipartite graphs $K(n, m)$ are χ-unique when $n \geq m \geq 2$.*

In 1988, Chia, Goh and Koh first investigated the chromaticity of complete tripartite graphs and obtained the following chromatically unique families of complete tripartite graphs.

Theorem 1.4.4. *([18])*
(i) $K(n,n,n+k)$*, for* $n \geq 2$ *and* $0 \leq k \leq 3$*, is* χ*-unique;*
(ii) $K(n-k,n,n)$*, for* $n \geq k+2$ *and* $0 \leq k \leq 3$*, is* χ*-unique;*
(iii) $K(n-k,n,n+k)$*, for* $n \geq 5$ *and* $0 \leq k \leq 2$*, is* χ*-unique.*

Chia, Goh and Koh [18] and Koh and Teo [43] proposed the following conjecture.

Conjecture 1.4.3. *([18, 43]) For any integers* n *and* k *with* $n \geq k+2 \geq 4$*,* $K(n-k,n,n)$ *is* χ*-unique.*

From 1998 to 2002, Zou and Shi improved the above results and gave the following results.

Theorem 1.4.5. *([83-87])*
(i) For $n > k + k^2/3$*,* $K(n-k,n,n)$ *is* χ*-unique graph;*
(ii) For $n > (k+k^2)/3$*,* $K(n,n,n+k)$ *is* χ*-unique graph;*
(iii) For $n > k^2 + \frac{2\sqrt{3}}{3}k$*,* $K(n-k,n,n+k)$ *is* χ*-unique graph;*
(iv) For $n \geq 6$*,* $K(n-4,n,n)$ *is* χ*-unique graph.*

In Section 5.3, we give a positive answer to Conjecture 1.4.3 and some general results.

Theorem 1.4.6. *For any integers* $n \geq m \geq r \geq 2$*, we have* $[K(r,m,n)] \subseteq \{K(x,y,z) - S | 1 \leq x \leq y \leq z, m \leq z \leq n, x+y+z = n+m+r, S \subset E(K(x,y,z))$ *and* $|S| = xy+xz+yz-nm-nr-mr\}$*. In particular, if* $z = n$*,* $K(r,m,n) = K(x,y,z)$*.*

Theorem 1.4.7. *For any integers* n *and* k *with* $n \geq k+2 \geq 4$*,* $K(n-k,n,n)$ *is* χ*-unique.*

Theorem 1.4.8. *For any integers* n *and* k *with* $n \geq 2k \geq 4$*,* $K(n-k,n-1,n)$ *is* χ*-unique.*

As a generalization of the above results, we get some general results in Section 5.4.

Theorem 1.4.9. *Let* $2 \le n_1 \le n_2 \cdots \le n_t$ *and* $G = K(n_1, n_2, \cdots, n_t)$. *If* $H \sim G$, *then*

(i) $H \in [G] \subset \{K(x_1, x_2, \cdots, x_t) - S | 1 \le x_1 \le x_2 \cdots \le x_t \le n_t, \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} n_i, S \subset E(K(x_1, x_2, \cdots, x_t))\}$;

(ii) there exists an integer $b \ge 2$ *such that* $x_1 \le x_2 \cdots \le x_b \le n_b - 1$ *and* K_{n_i} *is a component of* $\overline{H}$ *for any* $i \ge b+1$;

(iii) if $x_i = n_i$, *for any* $i \ge 3$, *then* $G = H$.

Theorem 1.4.10. *For any positive integers* $n \ge k+2$, $k \ge 2$ *and* $t \ge 3$, *the complete* t*-partite graph* $K(n-k, n, n, \cdots, n)$ *is* χ*-unique.*

Theorem 1.4.11. *For any positive integers* $n \ge 2k$, $k \ge 2$ *and* $t \ge 3$, *the complete* t*-partite graph* $K(n-k, n-1, n, \cdots, n)$ *is* χ*-unique.*

In 1988, Giudici and Lopez proved

Theorem 1.4.12. *([41]) The complete* t*-partite graph* $K(n-1, n, \cdots, n, n+1)$ *is* χ*-unique when* $t \ge 2$ *and* $n \ge 3$.

In 1990, Li and Liu proved

Theorem 1.4.13. *([48])* $K(1, n_2, \cdots, n_t)$ *is* χ*-unique if and only if* $max\{n_i | i = 2, 3, \cdots, t\} \le 2$.

Koh and Teo proposed the following problem in 1990.

Problem 1.4.1. *([43]) Let* $t \ge 2$. *Is the graph* $K(n_1, n_2, \cdots, n_t)$ χ*-unique if* $|n_i - n_j| \le 2$, *for all* $i, j = 1, 2, \cdots, t$, *and sufficiently large* $min\{n_1, n_2, \cdots, n_t\}$*?*

Very recently, Zou gave a partial answer to Problem 1.4.1 by the following theorem.

Theorem 1.4.14. *([88]) Let* $n_i \ge 2$, *for each* i, *and* $a_t = \sqrt{\frac{\sum_{1 \le i < j \le t} (n_i - n_j)^2}{2t}}$. *The complete* t*-partite graph* $K(n_1, n_2, \cdots, n_t)$ *is* χ*-unique if* $\sum_{i=1}^{t} n_i > t a_t^2 + \sqrt{2t(t-1)} a_t$ *and* $n_1, n_2, \cdots, n_t$ *satisfy one of the following conditions:*

(i) $n_1 = n_2 = \cdots = n_t$; *(ii)* $n_1 < n_2 \cdots < n_t$; *(iii)* $t = 3$ *or* 4.

In Section 5.5, we investigate the chromatic uniqueness of $K(n_1, n_2, \cdots, n_t)$. We solve Problem 1.4.1 by giving it a positive answer. Indeed, the following results solve Problem 1.4.1 and answer even more.

Theorem 1.4.15. *Let* $G = K(n_1, n_2, \cdots, n_t)$ *and* $n = \sum_{i=1}^{t} n_i$. *If* $n \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}$, *then* G *is* χ*-unique.*

Theorem 1.4.16. *Let* $G = K(n_1, n_2, \cdots, n_t)$. *If* $min\{n_i | i = 1, 2, \cdots, t\} \geq \sum\limits_{1\leq i<j\leq t} \frac{(n_i-n_j)^2}{2t} + \frac{\sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i-n_j)^2}}{t} + 1$, *then* G *is* χ*-unique.*

Theorem 1.4.17. *If* $|n_i - n_j| \leq k$ *and* $min\{n_1, n_2, \cdots, n_t\} \geq \frac{tk^2}{4} + \frac{\sqrt{2(t-1)}}{2}k + 1$, *then* $K(n_1, n_2, \cdots, n_t)$ *is* χ*-unique.*

Theorem 1.4.18. *If* $|n_i - n_j| = 2$ *and* $min\{n_1, n_2, \cdots, n_t\} \geq t + 1$, *then* $K(n_1, n_2, \cdots, n_t)$ *is* χ*-unique, where* $t \geq 2$.

We now turn our attention to general multipartite graphs. Let S be a set of s edges of G. Denote by $G - S$(or simply by $G - s$) the graph obtained from G by deleting all edges in S. We denote by $< S >$ the subgraph of G induced by S. Let S' be a set of s' edges of $\overline{G}$. Denote by $G + S'$ (or simply by $G + s'$) the graph obtained from G by adding all edges in S' to G. In particular, for $G = K(n_1, n_2, \cdots, n_t)$, we denote by $\mathcal{G}^{-s}_{n_1,n_2,\cdots,n_t}$ the family $\{G - S | S \subset E(G) \text{ and } |S| = s\}$. Let $G = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph with partition sets A_i such that $|A_i| = n_i$, where $i = 1, 2, \cdots, t$. By $K(A_i, A_j)$ we denote the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j = 1, 2, \cdots, t$. Denote by $K^{-K_{1,s}}_{i,j}(n_1, n_2, \cdots, n_t)$ the graph obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting all s edges of $K_{1,s}$ from $K(A_i, A_j)$ with center in A_i and others in A_j. Denote by $K^{-sK_2}_{i,j}(n_1, n_2, \cdots, n_t)$ the graph obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting all s edges of sK_2 from $K(A_i, A_j)$.

In 1986, Salzberg, López and Giudici investigated the chromatic uniqueness of the bipartite graphs and obtained the following.

Theorem 1.4.19. *([69]) Let* G *be the graph obtained from* $K(n, m)$ *by removing an edge. Then* G *is* χ*-unique when* $m \geq 3$ *and* $0 \leq n - m \leq 1$.

In 1988, Read proposed the following conjecture.

Conjecture 1.4.4. *([67]) All graphs obtained from $K(n,m)$, where $n,m \geq 3$, by removing an edge are χ-unique.*

In 1990, Teo and Koh confirmed this conjecture and proposed two new problems.

Theorem 1.4.20. *([71]) All graphs obtained from $K(n,m)$, where $n,m \geq 3$, by removing an edge are χ-unique.*

Problem 1.4.2. *([71]) For $n \geq m \geq 4$, study the chromaticity of $K(n,m)-2$.*

Problem 1.4.3. *([71]) For $n \geq m \geq 2$, study the chromaticity of $K(n,m)+1$.*

Later, several researchers studied the chromaticity of graphs obtained from $K(n,m)$ by deleting $s \geq 2$ edges and obtained some new results, see [15, 40, 42, 64, 78]. Recently, Dong, Koh, Teo, Little and Hendy obtained more general results and solved Problem 1.4.2, see [24, 25, 26].

Apart from the bipartite case, there are few known results on the chromaticity of general multipartite graphs. In 1988, Chia, Goh and Koh obtained

Theorem 1.4.21. *([18]) For $n \geq 2$ and $m \geq 4$, graphs obtained from $K(n,n,n)$, $K(n,n+1,n+1)$ or $K(m-1,m,m-1)$ by removing one edge are χ-unique.*

Let $< S >$ be a subgraph of K_n. We denote by $K^{+s}(n,n)$ the graph obtained from $K(n,n)$ by adding all edges in S between vertices of one of the partition sets in $K(n,n)$. For disjoint graphs G and H, $G+H$ denotes the *join graph* of G and H with vertex set $V(G) \cup V(H)$ and with edge set $\{xy|x \in V(G) \text{ and } y \in V(H)\} \cup E(G) \cup E(H)$. In Sections 6.3 and 6.4, we investigate the chromaticity of graphs $K^{+s}(n,n)$ and of graphs $K(n,m,r)-s$. We give a partial answer to Problem 1.4.3. Indeed, in the sections we prove the following results.

Theorem 1.4.22. *For $n \geq s+2$ and $s \geq 1$, let S be a set of s edges in K_n and let $< S >$ be a bipartite graph. Then $[K^{+s}(n,n)] = \{O_n + G | G \in [\overline{K_n - s}]\}$ and $K^{+s}(n,n)$ is χ-unique if and only if $\overline{K_n - s}$ is χ-unique, where $O_n + G$ is the join graph of O_n and G.*

Theorem 1.4.23. *For $n \geq s+2$ and $s \geq 1$, let S be a set of s edges in K_n and let $< S >$ be a bipartite graph. Then $K^{+s}(n,n)$ is χ-unique if and only if $< S >$ is a χ-unique graph without cut-vertex.*

Theorem 1.4.24. *Let $n_1 \leq n_2 \leq n_3$ with $n = n_1+n_2+n_3$ and $s \geq 1$. If $n > \frac{1}{2}\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2 + \sqrt{2\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2+12s}+3s+3$, then $\mathcal{G}^{-s}_{n_1,n_2,n_3}$ is χ-closed.*

Theorem 1.4.25. *Let $n_1+n_2+n_3 = n$ and $s \geq 1$. If $n > \frac{1}{2}\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2+\sqrt{2\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2+12s}+3s+3$, then $K^{-K_{1,s}}_{i,j}(n_1,n_2,n_3)$ is χ-unique, where $i \neq j$ and $i,j = 1,2,3$.*

Let $G = K(n_1,n_2,n_3)$ with $n_1 \leq n_2 \leq n_3$ and let A_1, A_2 and A_3 be three partition sets with $|A_i| = n_i$, where $i = 1,2,3$. We denote by $H^{-sK_2}_{n_1,n_2,n_3}$ the graph obtained by deleting all edges of sK_2 from $K(A_1,A_2)$ in G.

Theorem 1.4.26. *Suppose $n_1+n_2+n_3 = n$ and $s \geq 1$. If $n_1 \leq n_2 < n_3$ and $n > \frac{1}{2}\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2 + \sqrt{2\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2+12s}+3s+3$, then $H^{-sK_2}_{n_1,n_2,n_3}$ is χ-unique.*

In Sections 6.5 and 6.6, we investigate the chromaticity of general multipartite graphs. Many results are obtained on the chromaticity of general multipartite graphs. We list the main results in the following.

Theorem 1.4.27. *Let $G = K(n_1,n_2,n_3,n_4)$ and $S \subset E(G)$ such that $n = n_1+n_2+n_3+n_4$ and $|S| = s \geq 1$. If $n > \sqrt{3\sum_{1\leq i<j\leq 4}(n_i-n_j)^2+24s} + 4q(T_{n,4}) - 4q(G)+4s+4$, then $\mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$ is χ-closed.*

Theorem 1.4.28. *Let $G = K(n_1,n_2,n_3,n_4)$ and $n = n_1+n_2+n_3+n_4$. Suppose that $n_1 \leq n_2 \leq n_3 \leq n_4$, $s \geq 1$ and $n > \sqrt{3\sum_{1\leq i<j\leq 4}(n_i-n_j)^2+24s} + 4q(T_{n,4}) - 4q(G)+4s+4$. Then*
(i) every $K^{-K_{1,s}}_{i,j}(n_1,n_2,n_3,n_4)$ is χ-unique, for any (i,j) if $n_2+n_3 \neq n_1+n_4$,

where $i \neq j$ and $i, j = 1, 2, 3, 4$;

(ii) every $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique, for any (i, j) if $n_1 = n_2$ and $n_3 = n_4$, where $i \neq j$ and $i, j = 1, 2, 3, 4$;

(iii) every $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique, for $(i, j) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 4), (4, 2), (3, 4), (4, 3)\}$.

Theorem 1.4.29. *Let $G = K(n_1, n_2, n_3, n_4)$ with $n_1 \leq n_2 < n_3 \leq n_4$ and let $s \geq 1$. If $n > \sqrt{3\sum_{1\leq i<j\leq 4}(n_i - n_j)^2 + 24s} + 4q(T_{n,4}) - 4q(G) + 4s + 4$, then $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$ is χ-unique, where $n = n_1 + n_2 + n_3 + n_4$.*

For convenience, we replace $\overbrace{n, n, \cdots, n}^{a}$ by $a \times n$. For example, $K(t_1 \times n, t_2 \times (n+1))$ denotes the graph $K(\overbrace{n, n, \cdots, n}^{t_1}, \overbrace{n+1, n+1, \cdots, n+1}^{t_2})$ and $\mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1))$ denotes the family $\{K(\overbrace{n, n, \cdots, n}^{t_1}, \overbrace{n+1, n+1, \cdots, n+1}^{t_2}) - s | s \geq 1\}$. Suppose that $K(t_1 \times n, t_2 \times (n+1))$ has t partition sets A_i such that $|A_i| = n$ for $1 \leq i \leq t_1$ and $|A_i| = n+1$ for $t_1 + 1 \leq i \leq t$. We denote $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$ for $|A_i| = |A_j| = n$, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$ for $|A_i| = |A_j| = n+1$, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$ for $|A_i| = n$ and $|A_j| = n+1$ and $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$ for $|A_i| = n+1$ and $|A_j| = n$, respectively, by $H^{-K_{1,s}}(n, n)$, $H^{-K_{1,s}}(n+1, n+1)$, $H^{-K_{1,s}}(n, n+1)$ and $H^{-K_{1,s}}(n+1, n)$. Let $\mathcal{H}^{-K_{1,s}} = \{H^{-K_{1,s}}(n, n), H^{-K_{1,s}}(n+1, n+1), H^{-K_{1,s}}(n, n+1), H^{-K_{1,s}}(n+1, n)\}$.

Theorem 1.4.30. *Let $s \geq 1$, $n \geq 2$ and $t_1 \geq 1$. If $n \geq s + 2$, then $\mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1))$ is χ-closed.*

Theorem 1.4.31. *Let $G \in \mathcal{H}^{-K_{1,s}}$. If $s \geq 1$ and $n \geq s + 2$, then G is χ-unique.*

Let $K(n, n, (t-2) \times (n+1))$ have t partition sets A_i such that $|A_1| = |A_2| = n$ and $|A_i| = n + 1$ for $3 \leq i \leq t$.

Theorem 1.4.32. *Let $G = K(n, n, (t-2) \times (n+1))$. If $s \geq 1$ and $n \geq s + 2$, then $K_{1,2}^{-sK_2}(n, n, (t-2) \times (n+1))$ is χ-unique.*

In order to easily find the proofs of the new theorems mentioned in this introduction, we list here the corresponding theorems in the following chapters.

Theorems of Chapter 1	Corresponding Theorems
Theorem 1.2.6	Theorem 2.2.1
Theorem 1.2.8	Theorem 2.2.2
Theorem 1.2.9	Theorem 2.2.3
Theorem 1.2.10	Theorem 2.2.4
Theorem 1.2.11	Theorem 2.3.1
Theorem 1.2.12	Theorem 2.3.3
Corollary 1.2.2	Corollary 2.3.3
Theorem 1.2.14	Theorem 3.2.2
Theorem 1.2.15	Theorem 3.3.2
Theorem 1.2.16	Theorem 3.4.1
Corollary 1.2.3	Corollary 3.4.1
Theorem 1.2.17	Theorem 3.5.3
Theorem 1.2.18	Theorem 3.5.4
Theorem 1.3.7	Theorem 4.3.3
Theorem 1.3.8	Theorem 4.3.4
Theorem 1.3.9	Theorem 4.4.1
Theorem 1.3.14	Theorem 4.5.1
Corollary 1.3.1	Corollary 4.5.1
Theorem 1.3.15	Theorem 4.5.2
Corollary 1.3.2	Corollary 4.5.2
Theorem 1.3.16	Theorem 4.6.3
Theorem 1.3.17	Theorem 4.6.4
Theorem 1.3.18	Theorem 4.7.1
Theorem 1.3.19	Theorem 4.7.2
Theorem 1.3.20	Theorem 4.7.4
Theorem 1.3.21	Theorem 4.7.5

Theorems of Chapter 1	Corresponding Theorems
Theorem 1.3.22	Theorem 4.7.6
Theorem 1.4.6	Theorem 5.3.1
Theorem 1.4.7	Theorem 5.3.2
Theorem 1.4.8	Theorem 5.3.3
Theorem 1.4.9	Theorem 5.4.1
Theorem 1.4.10	Theorem 5.4.2
Theorem 1.4.11	Theorem 5.4.3
Theorem 1.4.15	Theorem 5.5.2
Theorem 1.4.16	Theorem 5.5.3
Theorem 1.4.17	Theorem 5.5.4
Theorem 1.4.18	Theorem 5.5.5
Theorem 1.4.22	Theorem 6.3.1
Theorem 1.4.23	Theorem 6.3.2
Theorem 1.4.24	Theorem 6.4.2
Theorem 1.4.25	Theorem 6.4.3
Theorem 1.4.26	Theorem 6.4.4
Theorem 1.4.27	Theorem 6.5.2
Theorem 1.4.28	Theorem 6.5.3
Theorem 1.4.29	Theorem 6.5.4
Theorem 1.4.30	Theorem 6.6.2
Theorem 1.4.31	Theorem 6.6.3
Theorem 1.4.32	Theorem 6.6.4

Chapter 2

Some Properties of Adjoint Polynomials of Graphs

2.1 Introduction

We recall the definition of adjoint polynomial of a graph and some of its important properties. For a graph G with p vertices, if G_0 is a spanning subgraph of G and each component of G_0 is a complete graph, then G_0 is called *an ideal subgraph* of G. Let $b_i(G)$ denote the number of ideal subgraphs of G with $p-i$ components. Then the following polynomial

$$h(G,x) = \sum_{i=0}^{p-1} b_i(G)x^{p-i}$$

is called *the adjoint polynomial* of G.

Example 2.1.1. *For C_5, we have $b_0(C_5) = 1, b_1(C_5) = 5, b_2(C_5) = 5$ and $b_3(C_5) = b_4(C_5) = 0$. So, $h(C_5, x) = x^5 + 5x^4 + 5x^3$.*

In 1987, Liu introduced the definition of adjoint polynomials of graphs and gave some useful properties.

Theorem 2.1.1. *([50]) Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G,x) = \prod_{i=1}^{k} h(G_i, x).$$

For an edge $e = v_1v_2$ of a graph G, the graph $G * e$ is defined as follows: the vertex set of $G * e$ is $(V(G)\backslash\{v_1, v_2\}) \cup \{v\}$, and the edge set of $G * e$ is $\{e'|e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv|u \in N_G(v_1) \cap N_G(v_2)\}$. For example, let e_1 be an edge of C_4 and e_2 an edge of K_4, then $C_4 * e_1 = P_2 \cup K_1$ and $K_4 * e_2 = K_3$.

Theorem 2.1.2. *([51]) Let G be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x).$$

In particular, if $e = u_1u_2$ does not belong to any triangle of G, then

$$h(G, x) = h(G - e, x) + xh(G - \{u_1, u_2\}, x).$$

Example 2.1.2. *For D_5 (see Figure 1.1), by Theorem 2.1.2 it is easy to get that $h(D_5, x) = x^5 + 5x^4 + 5x^3 + x^2$.*

Example 2.1.3. *For $G = C_5 \cup D_5$, by Theorem 2.1.1 and the above examples we have*

$$h(G, x) = h(C_5, x)h(D_5, x) = x^{10} + 10x^9 + 35x^8 + 51x^7 + 30x^6 + 5x^5.$$

From equations (1.1) and (1.2), one sees that it is not hard to get the chromatic polynomials of $\overline{C_5}$, $\overline{D_5}$ and $\overline{C_5 \cup D_5}$. In fact, it is easy to compute the chromatic polynomial of a dense graph by computing the adjoint polynomial of its complement. So, many researchers studied the properties of adjoint polynomials. Some useful and interesting properties were found by Dong, Teo, Little and Hendy [27, 28, 29], Du [31, 32], Liu [50-58], Ma [63], Wang and Liu [73, 74, 75], Ye and Li [79] and Zhao, Hou and Liu [80, 81, 82].

In the second section of this chapter, our main aim is to investigate recursive relations and divisibility of adjoint polynomials of some family of graphs. As an application of the recursive relations of adjoint polynomials, in Section 2.3 we study uniquely colorable graphs.

For a graph G, we denote by $t(G)$ the lowest term of $h(G, x)$ and by $\ell(G)$ the degree of $t(G)$, i.e., $\ell(G) = \partial(t(G))$. We denote by $h_1(G, x)$ the polynomial with a nonzero constant term such that $h(G, x) = x^{\ell(G)}h_1(G, x)$. For convenience, we denote $h(G, x)$ by $h(G)$ and $h_1(G, x)$ by $h_1(G)$.

2.2 The divisibility of adjoint polynomials of some graphs

Let G be a graph. For $w \in V(G)$ and $e \in E(G)$, by $G_w(P_m)$ (respectively $G_e(P_m)$) we denote the graph obtained from G and P_m by identifying a vertex w of G with an end-vertex of P_m (respectively by replacing the edge e of G by P_m).

Lemma 2.2.1. *If $m \geq 3$, then*

$$h(G_w(P_m)) = x[h(G_w(P_{m-1})) + h(G_w(P_{m-2}))].$$

Proof. Let uv be a pendant edge of $G_w(P_m)$ with $uv \in E(P_m)$. By Theorem 2.1.2, we have

$$\begin{aligned} h(G_w(P_m)) &= h(G_w(P_{m-1}))h(K_1) + xh(G_w(P_m - \{u,v\})) \\ &= x[h(G_w(P_{m-1})) + h(G_w(P_{m-2}))]. \end{aligned}$$

□

Theorem 2.2.1. *If $m \geq 4$, then*

$$h(G_e(P_m)) = x[h(G_e(P_{m-1})) + h(G_e(P_{m-2}))].$$

Proof. Let $e = uv \in E(G)$ and $u, v \in V(G_e(P_m))$. Take $H = G - e$ and $F = G - u$. Choose a vertex $w \in V(P_m)$ such that $uw \in E(G_e(P_m))$. When $m \geq 4$, by Theorem 2.1.2 we have

$$h(G_e(P_m)) = h(G_e(P_m) - uw) + xh(G_e(P_m) - \{u, w\}).$$

Note that $G_e(P_m) - uw = H_v(P_{m-1})$ and $G_e(P_m) - \{u,w\} = F_v(P_{m-2})$. By Lemma 2.2.1, it follows that

$$\begin{aligned} h(G_e(P_m)) = {} & x[h(H_v(P_{m-2})) + h(H_v(P_{m-3}))] \\ & +x^2[h(F_v(P_{m-3})) + h(F_v(P_{m-4}))] \\ = {} & x[h(H_v(P_{m-2})) + xh(F_v(P_{m-3}))] \\ & +x[h(H_v(P_{m-3})) + xh(F_v(P_{m-4}))] \\ = {} & x[h(G_e(P_{m-1})) + h(G_e(P_{m-2}))]. \end{aligned}$$

□

By Theorem 2.2.1, we can easily prove the following.

Lemma 2.2.2. *([58, 62]) (i) For $n \geq 3$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$;*
(ii) For $n \geq 6$, $h(C_n) = x(h(C_{n-1}) + h(C_{n-2}))$;
(iii) For $n \geq 6$, $h(D_n) = x(h(D_{n-1}) + h(D_{n-2}))$;
(iv) For $n \geq 8$, $h(F_n) = x(h(F_{n-1}) + h(F_{n-2}))$.

Theorem 2.2.2. *Let $\{g_i(x)\}_i$ $(i \geq 0)$ be a sequence of polynomials with integral coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*
(i) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$;
(ii) $h_1(P_n)|g_{n+1+i}(x)$ if and only if $h_1(P_n)|g_i(x)$, for any positive integers n and i.

Proof. (i) By induction on k. Since $h(P_1) = x$ and $h(P_2) = x^2+x$, by Lemma 2.2.2(i) we get $h(P_0) = 1$. Thus, we have

$$g_n(x) = h(P_1)g_{n-1}(x) + xh(P_0)g_{n-2}(x).$$

So, (i) holds when $k = 1$. Suppose that it is true for $k \leq l-1$. From the recursive relation of $g_n(x)$, Lemma 2.2.2(i) and the induction hypothesis, we have

$$\begin{aligned} g_n(x) &= x(g_{n-1}(x) + g_{n-2}(x)) \\ &= xh(P_{l-1})g_{n-l}(x) + x^2h(P_{l-2})g_{n-l-1}(x)+ \\ &\quad\; xh(P_{l-2})g_{n-l}(x) + x^2h(P_{l-3})g_{n-l-1}(x) \\ &= h(P_l)g_{n-l}(x) + xh(P_{l-1})g_{n-l-1}(x). \end{aligned}$$

(ii) From (i), for any integers n and i, it follows that

$$g_{n+1+i}(x) = h(P_{n+1})g_i(x) + xh(P_n)g_{i-1}(x).$$

It is not difficult to see that $(h_1(P_n), h_1(P_{n+1})) = 1$ and $(h_1(P_n), x) = 1$ for $n \geq 2$. So, from the above equality we have $h_1(P_n)|g_{n+1+i}(x)$ if and only if $h_1(P_n)|g_i(x)$. □

Remark 2.2.1. *From Theorem 2.2.2 (ii), we have that $h_1(P_n)|g_{(n+1)k+i}(x)$ if and only if $h_1(P_n)|g_{(n+1)(k-1)+i}(x)$, for $k \geq 1$. Take $m = (n+1)k+i$ and $0 \leq i \leq n$. Then it follows that $h_1(P_n)|g_m(x)$ if and only if $h_1(P_n)|g_i(x)$, where $0 \leq i \leq n$. The result is used in the proofs of Theorems 2.2.3, 2.2.4 and 2.2.5.*

From Theorem 1.2.4, we have

Lemma 2.2.3. *For $n \geq 2$, $\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor$ and $\ell(P_n) = \lfloor \frac{n+1}{2} \rfloor$.*

Lemma 2.2.4. *(i) For $t \geq 1$ and $m \geq 4$, $h(T_{1,t,m}) = x[h(T_{1,t,m-1}) + h(T_{1,t,m-2})]$; (ii) Let $n = |V(T_{1,t,m})| = m + t + 2$. Then*

$$\partial h_1(T_{1,t,m}) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } t \text{ and } m \text{ are even,} \\ \lfloor \frac{n-1}{2} \rfloor, & \text{otherwise.} \end{cases}$$

(iii) Let $n = |V(T_{1,t,m})| = m + t + 2$. Then

$$\ell(T_{1,t,m}) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } t \text{ and } m \text{ are even,} \\ \lfloor \frac{n+2}{2} \rfloor, & \text{otherwise.} \end{cases}$$

Proof. (i) This is obvious from Lemma 2.2.1.

(ii) Choose the edge $e = uv \in E(T_{1,t,m})$ such that $d_u = 1$ and $d_v = 3$. By Theorem 2.1.2, $h(T_{1,t,m}) = x[h(P_{m+t+1}) + h(P_t)h(P_m)]$. From Lemma 2.2.3, we have $\partial(h_1(P_{m+t+1})) = \lfloor \frac{m+t+1}{2} \rfloor$ and $\partial(h_1(P_t)h_1(P_m)) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{t}{2} \rfloor$. Clearly, $\partial(h_1(P_{m+t+1})) \geq \partial(h_1(P_t)h_1(P_m))$. Noticing that $\partial(h(P_{m+t+1})) = \partial(h(P_t)h(P_m)) + 1$, we have

$$\partial(h_1(T_{1,t,m})) = \partial(h_1(P_{m+t+1})) + 1 \text{ for } \partial(h_1(P_{m+t+1})) = \partial(h_1(P_t)h_1(P_m))$$

and

$$\partial(h_1(T_{1,t,m})) = \partial(h_1(P_{m+t+1})) \text{ for } \partial(h_1(P_{m+t+1})) > \partial(h_1(P_t)h_1(P_m)).$$

It is not difficult to verify that $\partial(h_1(P_{m+t+1})) = \partial(h_1(P_t)h_1(P_m))$ only if m and t are even. So, (ii) holds.

Clearly, (iii) follows from (ii). □

Lemma 2.2.5. *([56]) Let $1 \leq r_1 \leq r_2$ and $r_1 \leq s_1 \leq s_2$ such that $r_1 + r_2 = s_1 + s_2$. Then $h(P_{r_1})h(P_{r_2}) - h(P_{s_1})h(P_{s_2}) = (-1)^{r_1} x^{r_1+1} h(P_{s_1-r_1-1})h(P_{s_2-r_1-1})$, where $h(P_0) = 1$.*

Theorem 2.2.3. *For $k \geq 1$ and $t \geq 1$ such that $kt > 3$, we have that $h(P_{t-1}) | h(T_{1,t,kt-3})$, $h(P_t) | h(T_{1,t,kt+k-1})$ and $h(P_{t+2}) | h(T_{1,t,k(t+3)})$.*

Proof. Suppose that $g_0(x) = (-1)^t \frac{h(P_t)^2}{x^t}$, $g_1(x) = (-1)^{t-1} \frac{h(P_t)h(P_{t-3})+h(P_{t-1})^2}{x^{t-2}}$ and $g_n(x) = x[g_{n-1}(x) + g_{n-2}(x)]$. We have the following claim.

Claim. For $n \geq t+3$, $g_n(x) = h(T_{1,t,n-t-2})$.
Proof of the claim: Noticing that $h(P_t)^2 = x(h(P_t)h(P_{t-2}) + h(P_t)h(P_{t-1}))$, from Theorem 2.2.2 and Lemma 2.2.5, we can obtain by calculation that

$$\begin{aligned} g_{t+3}(x) &= h(P_{t+2})g_1(x) + xh(P_{t+1})g_0(x) \\ &= \tfrac{(-1)^{t-1}h(P_t)}{x^{t-2}}\left[h(P_{t-3})h(P_{t+2}) - h(P_{t-2})h(P_{t+1})\right] \\ &\quad + \tfrac{(-1)^{t-1}h(P_{t-1})}{x^{t-2}}\left[h(P_{t-1})h(P_{t+2}) - h(P_t)h(P_{t+1})\right] \\ &= h(P_3)h(P_t) + x^3h(P_{t-1}). \end{aligned}$$

By Theorem 2.1.2, $h(T_{1,t,1}) = h(P_3)h(P_t) + x^3h(P_{t-1})$. Thus, $g_{t+3}(x) = h(T_{1,t,1})$.

Similarly, from Theorems 2.1.1 and 2.1.2 and Lemma 2.2.5, we can show that $g_{t+4}(x) = h(T_{1,t,2}) = h(P_4)h(P_t) + x^2h(P_2)h(P_{t-1})$. Using the recursive relation of $g_n(x)$, from (i) of Lemma 2.2.4, we have $g_n(x) = h(T_{1,t,n-t-2})$ for $n \geq t+3$. This completes the proof of the claim.

Using the recursive relation of $g_n(x)$, from (i) of Theorem 2.2.2, we can obtain by calculation that $g_{t+2}(x) = \frac{g_{t+4}(x)-xg_{t+3}(x)}{x} = h(P_{t+2})$, $g_{t+1}(x) = \frac{g_{t+3}(x)-xg_{t+2}(x)}{x} = xh(P_t)$ and $g_{t-1}(x) = \frac{(x+1)g_{t+1}(x)-g_{t+2}(x)}{x} = xh(P_{t-1})$. Clearly, $h_1(P_{t-1})|g_{t-1}(x)$, $h_1(P_t)|g_{t+1}(x)$, $h_1(P_{t+2})|g_{t+2}(x)$. So, by (ii) of Theorem 2.2.2, $h_1(P_{t-1})|g_{kt+t-1}(x)$, $h_1(P_t)|g_{(t+1)k+t+1}(x)$ and $h_1(P_{t+2})|g_{(t+3)k+t+2}(x)$. Note that $g_n(x) = h(T_{1,t,n-t-2})$ for $n \geq t+3$. We have that $h_1(P_{t-1})|h(T_{1,t,kt-3})$, $h_1(P_t)|h(T_{1,t,kt+k-1})$ and $h_1(P_{t+2})|h(T_{1,t,k(t+3)})$. Thus, from Lemmas 2.2.3 and 2.2.4, it is not difficult to see that $h(P_{t-1})|h(T_{1,t,kt-3})$, $h(P_t)|h(T_{1,t,kt+k-1})$ and $h(P_{t+2})|h(T_{1,t,k(t+3)})$. □

Theorem 2.2.4. *For $l \geq 2$, $m \geq 1$ and $k \geq 1$, we have:*
(i) $h(P_l)|h(T_{1,1,m})$ if and only if $(l,m) \in \{(3,4k)\}$;
(ii) $h(P_l)|h(T_{1,2,m})$ if and only if $(l,m) \in \{(2,3k-1),(4,5k)\}$;
(iii) $h(P_l)|h(T_{1,3,m})$ if and only if $(l,m) \in \{(2,3k),(3,4k-1),(5,6k)\}$;
(iv) $h(P_l)|h(T_{1,4,m})$ if and only if $(l,m) \in \{(3,4k-3),(4,5k-1),(6,7k)\}$;
(v) $h(P_l)|h(T_{1,5,m})$ if and only if $(l,m) \in \{(2,3k-1),(3,4k),(4,5k-3),(5,6k-1),(7,8k)\}$;
(vi) $h(P_l)|h(T_{1,6,m})$ if and only if $(l,m) \in \{(2,3k),(5,6k-3),(6,7k-1),(8,9k)\}$.

Proof. Let $g_0(x) = (-1)^t \frac{h(P_t)^2}{x^t}$, $g_1(x) = (-1)^{t-1} \frac{h(P_t)h(P_{t-3})+h(P_{t-1})^2}{x^{t-2}}$ and $g_n(x) = x[g_{n-1}(x) + g_{n-2}(x)]$. From the proof of Theorem 2.2.3, one can see that, for $n \geq t+3$, $g_n(x) = h(T_{1,t,n-t-2})$.

Without loss of generality, assume that $n = (l+1)k + i$, where $0 \leq i \leq l$. By Theorem 2.2.2, see Remark 2.2.1, $h_1(P_l)|g_n(x)$ if and only if $h_1(P_l)|g_i(x)$ for $0 \leq i \leq l$. Note that $g_i(x) = h(T_{1,t,i-t-2})$, for $l \geq t+3$. From Lemma 2.2.3 and (ii) of Lemma 2.2.4, we have $\partial h_1(P_l) = \lfloor l/2 \rfloor$ and $\partial(g_i(x)) = \partial h_1(T_{1,t,i-t-2}) \leq \lfloor i/2 \rfloor \leq \lfloor l/2 \rfloor$. Thus, if $h_1(P_l)|h_1(T_{1,t,i-t-2})$, then $\partial(h_1(P_l)) = \partial(h_1(T_{1,t,i-t-2}))$. Moreover, it must hold that $h_1(P_l) = h_1(T_{1,t,i-t-2})$. So, by the definition of $R_1(G)$ (see the later Section 4.2), $R_1(P_l) = R_1(T_{1,t,i-t-2})$, which contradicts that $R_1(P_l) \neq R_1(T_{1,t,i-t-2})$. Therefore, we have that, for $l \geq t+3$, $h(P_l) \not| h(T_{1,t,i-t-2})$. Thus, it is sufficient to consider the cases in which $l \leq t+2$.

Case 1. $t = 1$. Clearly, $l \leq 3$.

By calculation we have that $g_0(x) = -x$, $g_1(x) = x$, $g_2(x) = x^2$ and $g_3(x) = h(P_3)$. It is easy to verify that $h_1(P_l)|g_i(x)$ if and only if $l = i = 3$ for $2 \leq l \leq 3$ and $0 \leq i \leq 3$. By Theorem 2.2.2(ii), $h_1(P_3)|g_{4k+3}(x)$. Thus, $h_1(P_l)|h(T_{1,1,m})$ if and only if $l = 3$ and $m = 4k$, where $k \geq 1$. From Lemma 2.2.3 and (iii) of Lemma 2.2.4, we can obtain that if $m \geq 4$, then $h(P_l)|h(T_{1,1,m})$ if and only if $l = 3$ and $m = 4k$, for $k \geq 1$. This completes the proof of (i).

Case 2. $t = 2$. So, $l \leq 4$.

By calculation, it is easy to obtain that $g_0(x) = [h_1(P_2)]^2$, $g_1(x) = -x^2$, $g_2(x) = 2x^2 + x$, $g_3(x) = x^2 h(P_2)$ and $g_4(x) = x^2 h_1(P_4)$. One can see that $h_1(P_l)|g_i(x)$ if and only if $(l, i) \in \{(2,0), (2,3), (4,4)\}$, for $2 \leq l \leq 4$ and $0 \leq i \leq 4$. From Theorem 2.2.2 (ii), it is not difficult to see that $h_1(P_2)|g_{3k+3}$ and $h_1(P_4)|g_{5k+4}$. Hence, $h_1(P_l)|h(T_{1,2,m})$ if and only if $(l, m) \in \{(2, 3k-1), (4, 5k)\}$. With a proof similar to that of (i), we know that (ii) holds.

Case 3. $t = 3$. So, $l \leq 5$.

By calculation, we have that $g_0(x) = -[h_1(P_3)]^2$, $g_1(x) = x(x^2 + 3x + 3)$, $g_2(x) = -x^2 h_1(P_2)$, $g_3(x) = x^2(2x+3)$, $g_4(x) = x^3 h_1(P_3)$ and $g_5(x) = x^3 h_1(P_5)$. One can verify that $h_1(P_l)|g_i(x)$ if and only if $(l, i) \in \{(3,0), (2,2), (3,4), (5,5)\}$, for $2 \leq l \leq 4$ and $0 \leq i \leq 5$. With a proof completely similar to that of (i), we can show that (iii) holds.

Similarly, we can show that (iv), (v) and (vi) hold. Here we only give the expression of $g_i(x)$. The details of the proof are omitted.

When $t=4$, $g_0(x)=[h_1(P_4)]^2$, $g_1(x)=-x(x^3+5x^2+7x+1)$, $g_2(x)=x(x^3+4x^2+5x+1)$, $g_3(x)=-x^3h_1(P_3)$, $g_4(x)=x^2(2x^2+5x+1)$, $g_5(x)=x^3h_1(P_4)$ and $g_6(x)=x^3h_1(P_6)$.

When $t=5$, $g_0(x)=-[h_1(P_5)]^2$, $g_1(x)=x(x^4+7x^3+16x^2+13x+4)$, $g_2(x)=-x^2(x^3+6x^2+11x+5)$, $g_3(x)=x^2h_1(P_2)h_1(P_3)$, $g_4(x)=-x^3h_1(P_4)$, $g_5(x)=x^3(2x^2+7x+4)$, $g_6(x)=x^4h_1(P_5)$ and $g_7(x)=x^4h_1(P_7)$.

When $t=6$, $g_0(x)=[h_1(P_6)]^2$, $g_1(x)=-x(x^5+9x^4+29x^3+40x^2+22x+2)$, $g_2(x)=xh_1(P_2)(x^4+7x^3+15x^2+9x+1)$, $g_3(x)=-x(x^4+7x^3+16x^2+12x+1)$, $g_4(x)=x^2(x^4+6x^3+12x^2+9x+1)$, $g_5(x)=-x^4h_1(P_5)$, $g_6(x)=x^3(2x^3+9x^2+9x+1)$, $g_7(x)=-x^4h_1(P_6)$ and $g_8(x)=x^4h_1(P_8)$.

The proof of the theorem is now complete. □

From Theorem 2.2.4, it is not difficult to see that, for $1\le t\le 6$ and $n\ge 2$, $h(P_n)|h(T_{1,t,m})$ if and only if $n+1|t$, or $n+1|t+1$, or $n+1|t+3$. So, we propose the following problem.

Problem 2.2.1. *For $n\ge 2$ and $m\ge t\ge 1$, find a necessary and sufficient condition for $h(P_n)|h(T_{1,t,m})$. In particular, is it true that $h(P_n)|h(T_{1,t,m})$ if and only if $n+1|t$, or $n+1|t+1$, or $n+1|t+3$?*

By Theorem 2.2.2 and Lemma 2.2.2, it is not hard to obtain a sufficient and necessary condition for $h(P_n)|h(H)$, where $H\in\{P_n,C_n,D_n,F_n,A_n,B_n\}$.

Theorem 2.2.5. *For $n\ge 2$, we have*
(i) $h(P_n)|h(P_m)$ if and only if $(n+1)|(m+1)$;
(ii) $h(P_n)|h(C_m)$ if and only if $n=3$ and $m=4k+2$;
(iii) $h(P_n)|h(D_m)$ if and only if $n=2$ and $m=3k+2$, or $n=4$ and $m=5k+3$;
(iv) $h(P_n)|h(F_m)$ if and only if $n=2$ and $m=3k+2$, or $n=4$ and $m=5k+2$;
(v) $h(P_n)|h(A_m)$ if and only if $n=2$ and $m=3k+2$;
(vi) $h(P_n)\nmid h(B_m)$ for all $n\ge 2$ and $m\ge 6$.

2.3 Adjoint polynomials of graphs and uniquely colorable graphs

Let λ be a positive integer. A λ-coloring of G is a partition of $V(G)$ into λ color classes such that the vertices in the same color class are not adjacent. If every $\chi(G)$–coloring of G gives the same partition of $V(G)$, then G is said to be *a uniquely $\chi(G)$–colorable graph*.

For a graph G with p vertices, we denote by $\alpha(G, r)$ the number of r-independent partitions of $V(G)$, where $r = 1, 2, \cdots, p$. Clearly, G is uniquely $\chi(G)$-colorable if and only if $\alpha(G, \chi(G)) = 1$. Let $h(G) = \sum_{i=0}^{p} b_i(G)x^{p-i}$. Recalling that $b_i(G) = \alpha(\overline{G}, p-i)$, we have that $\ell(G) = \chi(\overline{G})$ and $\alpha(\overline{G}, \chi(\overline{G}))$ is the coefficient of $t(G)$. Thus, the following basic result follows.

Lemma 2.3.1. *Let G be a graph. Then $\overline{G}$ is uniquely n-colorable if and only if $t(G) = x^n$.*

In [12, 13, 14], the unique n-colorability of graphs was studied. Some results of the unique n-colorability of graphs were obtained. In this section, we obtain some new results on the unique colorability of graphs by using properties of the adjoint polynomials of graphs. Furthermore, we generalize the results in [14].

Lemma 2.3.2. *Let m be a positive integer. Then $t(K_m) = x$.*

Definition 2.3.1. *([33]) Let G be a graph with vertex v, and $N_G(v) = A \cup B$ and $A \cap B = \emptyset$. Then $H = (G, v, A, B)$ is the graph defined as follows: $V(H) = (V(G) - \{v\}) \cup \{v_1, v_2\}(v_1, v_2 \notin V(G))$ and $E(H) = \{e \in E(G) | e$ is not incident with $v\} \cup \{v_1u | u \in A\} \cup \{v_2u | u \in B\}$. H is called the graph obtained from G by splitting vertex v, and we write $H = G|_v$. H is said to be a vertex splitting graph of G if H is obtained from G by a sequence of vertex splitting.*

Definition 2.3.2. *([33]) For a graph G with $A, B \subseteq V(G)$, A and B are said to be adjacent in G if for any $x \in A$ and $y \in B$, we have $xy \in E(G)$.*

Lemma 2.3.3. *([33]) Let G be a graph with vertex v and $H = (G, v, A, B)$. Then $h_1(H, x) = h_1(G, x)$ if and only if A and B are adjacent in G.*

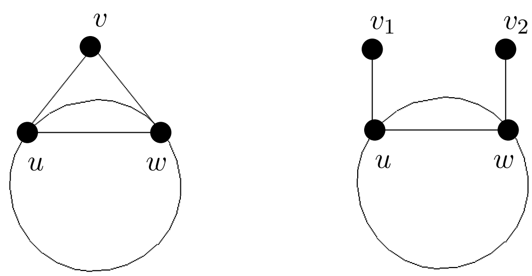


Figure 2.1

We consider a graph G containing K_3 as a subgraph. Let $\{u,v,w\} = V(K_3) \subset V(G)$ and $d_G(v) = 2$. Let $A = \{u\}$ and $B = \{w\}$. It is clear that A and B are adjacent in G, see Figure 2.1. By Lemma 2.3.3, we have $h_1(G,x) = h_1(G|_v,x)$.

Denote by $G(K_m,P_s)$ the graph obtained from K_m and P_s by identifying a vertex of K_m with a vertex of degree 1 of P_s. Clearly $G(K_m,P_1) = K_m$.

Lemma 2.3.4. *Let $s \geq 3$. Then*
(i) $h(G(K_m,P_s),x) = x(h(G(K_m,P_{s-1}),x) + h(G(K_m,P_{s-2}),x))$;
(ii) $t(G(K_m,P_s)) = \begin{cases} x^{\frac{s+1}{2}}, & \text{if } s \text{ is odd,} \\ \frac{s+2}{2}x^{\frac{s+2}{2}}, & \text{if } s \text{ is even.} \end{cases}$

Proof: (i) This is obvious from Lemma 2.2.1.

(ii) By induction on s.

From Lemma 2.2.1, we have that $h(G(K_m,P_2),x) = x(h(K_m,x)+h(K_{m-1},x))$ and $h(G(K_m,P_3),x) = x(h(G(K_m,P_2),x) + h(K_m,x))$. So, by Lemma 2.3.2, we get that

$$t(G(K_m,P_2)) = 2x^2 \text{ and } t(G(K_m,P_3)) = x^2.$$

Suppose that (ii) holds when $s < k$, where $k \geq 4$. By Lemma 2.2.1,

$$h(G(K_m,P_k),x) = x(h(G(K_m,P_{k-1}),x) + h(G(K_m,P_{k-2}),x)).$$

If k is even, then $t(G(K_m,P_{k-1})) = x^{\frac{k}{2}}$ and $t(G(K_m,P_{k-2})) = \frac{k}{2}x^{\frac{k}{2}}$, by the induction hypothesis. Hence $t(G(K_m,P_k)) = \frac{k+2}{2}x^{\frac{k+2}{2}}$.

If k is odd, then $t(G(K_m, P_{k-1})) = \frac{k+1}{2}x^{\frac{k+1}{2}}$ and $t(G(K_m, P_{k-2})) = x^{\frac{k-1}{2}}$, by the induction hypothesis. Therefore $t(G(K_m, P_k)) = x^{\frac{k+1}{2}}$. $\square$

By Lemmas 2.3.1 and 2.3.4, we prove easily the following results.

Theorem 2.3.1. *Let s be an odd integer. Then $\overline{G(K_m, P_s)}$ is a uniquely $\frac{s+1}{2}$–colorable graph with $m + s - 1$ vertices.*

Corollary 2.3.1. *([14]) For any $n \geq 1$ and $m \geq 2$, we have*
(i) $\overline{P_{2n}}$ is a uniquely n-colorable graph with $2n$ vertices;
(ii) $\overline{D_{2m+1}}$ is a uniquely m-colorable graph with $2m + 1$ vertices.

Theorem 2.3.2. *Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then $\overline{G}$ is uniquely n-colorable if and only if each complement $\overline{G_i}$ is uniquely m_i-colorable and $n = \sum\limits_{i=1}^{k} m_i$.*

Proof: By Theorem 2.1.1, $t(G) = \prod_{i=1}^{k} t(G_i)$. The theorem follows from Lemma 2.3.1. $\square$

By applying Theorems 2.3.1 and 2.3.2, we can find many families of uniquely n-colorable graphs with m vertices, where $n \geq 3$ and $m \geq 3$. So, we have the following corollary.

Corollary 2.3.2. *There exist infinitely many uniquely n-colorable graphs with m vertices, where $n \geq 3$, $m \geq 3$ and $m \geq n$.*

Let $v \in V(G(K_m, P_s))$ such that $v \in V(K_m)$ and $d(v) = m - 1$. By $G'(K_m, P_s)$ we denote the graph obtained from $G(K_m, P_s)$ by splitting the vertex v.

Theorem 2.3.3. *Let s be an odd integer. Then $\overline{G(K_m, P_s) \cup K_1}$ and $\overline{G'(K_m, P_s)}$ are uniquely $\frac{s+3}{2}$-colorable graphs that are chromatically equivalent.*

Proof: Choose A and B such that $A \cap B = \emptyset$ and $A \cup B = V(K_m) \backslash \{v\}$, where $A \neq \emptyset$ and $B \neq \emptyset$. Note that A and B are adjacent in $G(K_m, P_s)$. By Lemma 2.3.3, we have $h_1(G'(K_m, P_s), x) = h_1(G(K_m, P_s), x)$. Since

$$p(G'(K_m, P_s)) = p(G(K_m, P_s) \cup K_1),$$

we have

$$h(G'(K_m, P_s), x) = h(G(K_m, P_s) \cup K_1, x).$$

From Lemma 2.3.4,

$$t(G'(K_m, P_s)) = \begin{cases} x^{\frac{s+3}{2}}, & \text{if } s \text{ is odd,} \\ \frac{s+4}{2} x^{\frac{s+4}{2}}, & \text{if } s \text{ is even.} \end{cases}$$

The theorem now follows from Lemma 2.3.1. □

Note that there are many ways of choosing A and B such that A and B are adjacent in $G(K_m, P_s)$. There exist many graphs that are chromatically equivalent with $\overline{G(K_m, P_s) \cup K_1}$. Suppose that v is split into two vertices v_1 and v_2. If $d(v_1) \neq 1$ or $d(v_2) \neq 1$, then v_1 or v_2 can be split in $G'(K_m, P_s)$. Hence, we can obtain many graphs that are chromatically equivalent with $\overline{G(K_m, P_s) \cup 2K_1}$. Repeating the above procedure, by Theorem 2.3.2 we obtain the following corollary.

Corollary 2.3.3. *Let $n \geq 3$. There exist infinitely many uniquely n-colorable graphs that are chromatically equivalent.*

Remarks

In this chapter, we investigated recursive relations and divisibility of adjoint polynomials of some special graphs. Theorem 2.2.1 gave a class of graphs such that their adjoint polynomials satisfy the recursive relation $g_m(x) = x[g_{m-1}(x) + g_{m-2}(x)]$ and Theorem 2.2.2 gave a way to find a necessary and sufficient condition for $h_1(P_n)|g_m(x)$, for $n \geq 2$. We obtained some necessary and sufficient conditions for $h_1(P_n)|h(H)$, see Theorems 2.2.3, 2.2.4 and 2.2.5. As an application of the recursive relations of adjoint polynomials, some new uniquely colorable graphs were obtained in Section 2.3, see Theorems 2.3.1 and 2.3.3. The results in Section 2.2 will be used in Chapters 3 and 4, whereas the results in Section 2.3 do not play a role in later chapters.

Chapter 3

On the Roots of Adjoint Polynomials of Graphs

3.1 Introduction

Roots and properties of chromatic polynomials of graphs and the adjoint polynomials of their complements have been studied for several years. In particular, Brenti, Royle and Wagner [4, 5] investigated the roots and log-concavity of the coefficients of the adjoint polynomial of a graph. They showed that $h(G, x)$ has only real roots for many general classes of graphs, such as comparability graphs, triangle-free graphs and so on.

For a polynomial $f(x)$, if a root of $f(x)$ is not real, then the root is said to be an *unreal root*. In this chapter, we investigate the minimum real roots of adjoint polynomials and determine some classes of graphs with unreal roots. For a graph G, let $\beta(G)$ denote the minimum real roots of $h(G, x)$. In Section 3.2, we give some basic results on the minimum real roots of adjoint polynomials of some graphs. We determine in Sections 3.3 and 3.4 all connected graphs such that the minimum real roots of their adjoint polynomials belong to the interval $[-4, 0]$ and to the interval $[-(2+\sqrt{5}), -4)$, respectively. In Section 3.5, we give the way to construct graphs such that their σ-polynomials have at least one unreal root. A problem posed by Brenti, Royle and Wagner [5] in 1994 is solved as well.

For a graph G, let $f(G,x)$ denote the characteristic polynomial of G. We denote by $\rho(G)$ the maximum real roots of $f(G,x)$.

3.2 Some basic properties of the minimum real roots of an adjoint polynomial

In this section, we give some fundamental inequalities and equalities on the minimum real roots of the adjoint polynomial of G. The following result can be found in [74].

Theorem 3.2.1. *([74]) For a tree T, $\beta(T) = -(\rho(T))^2$.*

Lemma 3.2.1. *([81]) Let $f_1(x), f_2(x)$ and $f_3(x)$ be polynomials in x with real positive coefficients. If (i) $f_3(x) = f_2(x) + f_1(x)$ and $\partial f_3(x) - \partial f_1(x) \equiv 1(mod\ 2)$, (ii) both $f_1(x)$ and $f_2(x)$ have real roots, and $\beta_2 < \beta_1$, then $f_3(x)$ has at least one real root β_3 such that $\beta_3 < \beta_2$, where β_i denotes the minimum real roots of $f_i(x)$, where $i = 1, 2, 3$.*

Theorem 3.2.2. *Let G be a connected graph and let H be a proper subgraph of G. Then*

$$\beta(G) < \beta(H).$$

Proof. Let q be the number of edges of G. We prove the theorem by induction on q.

It is obvious that the result holds when $q = 1$.

Let G be a graph with $q \geq 2$ and suppose that the theorem holds when G has fewer than q edges. Since H is a proper subgraph of G, we can choose an edge e in G such that either H is a proper subgraph of $G-e$ or $H = G-e$. So, select the edge e in G such that H is a subgraph of $G-e$, then by Theorem 2.1.2 we have

$$h(G,x) = h(G-e,x) + h(G*e,x).$$

The graph $G-e$ has p vertices and $q-1$ edges, and $G*e$ has $p-1$ vertices and at most $q-2$ edges. Note that $G*e$ is a proper subgraph of $G-e$ and each connected component of $G*e$ is a proper subgraph of some connected component of $G - e$, if e is a cut-edge of G. By the induction hypothesis and Theorem 2.1.1, we have

$$\beta(G - e) < \beta(G * e).$$

Since $\partial h(G) = \partial h(G*e)+1$, from Lemma 3.2.1 we obtain that $\beta(G) < \beta(G-e)$. Note that H is a subgraph of $G - e$, then, by the induction hypothesis, we have $\beta(G - e) \leq \beta(H)$. So,

$$\beta(G) < \beta(G - e) \leq \beta(H).$$

□

From Theorem 1.2.5, $\beta(C_n) = \beta(T_{1,1,n-2})$ and $\beta(D_n) = \beta(T_{1,2,n-3})$, for $n \geq 4$. Then the following corollary follows from Theorem 3.2.2.

Corollary 3.2.1. *([75]) (i) For $n \geq 2$, $\beta(P_n) < \beta(P_{n-1})$;*
(ii) For $n \geq 4$, $\beta(C_n) < \beta(C_{n-1})$ and $\beta(D_{n+1}) < \beta(D_n)$;
(iii) For $n \geq 4$, $\beta(D_n) < \beta(C_n) < \beta(P_n)$.

An *internal x_1x_k-path* of a graph G is a path $x_1x_2x_3 \cdots x_k$(possibly $x_1 = x_k$) of G such that $d(x_1)$ and $d(x_k)$ are at least 3 and $d(x_2) = d(x_3) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$), where $d(x_i)$ denotes the degree of the vertex x_i in G.

Lemma 3.2.2. *([21]) Let G_{xy} be the graph obtained from G by inserting a new vertex on the edge xy of G. If xy is an edge on an internal path of G and $G \ncong U_n$, for all $n \geq 6$, then $\rho(G_{xy}) < \rho(G)$.*

Theorem 3.2.3. *Let G be a tree. If uv is an edge on an internal path of G and $G \ncong U_n$, for all $n \geq 6$, then $\beta(G) < \beta(G_{uv})$.*

Proof: The theorem follows directly from Theorem 3.2.1 and Lemma 3.2.2. □

From Lemma 2.3.3 and Figure 2.1, we have

Lemma 3.2.3. *Let G be a graph with a triangle uvw and $d_u(G) = 2$. Then*

$$xh(G, x) = h(G|_u, x).$$

For a graph G with a vertex u, the path tree $T(G,u)$ is defined as follows: $T(G,u)$ is the tree with the paths in G which start at u as its vertices, and where two such vertices are joined by an edge if one path is maximal subpath of the other. We call $T = T(G,u)$ *the path tree of G which starts at u*. In order to give some feeling about the construction of a path tree, we would like to give the following example, see Figure 3.1.

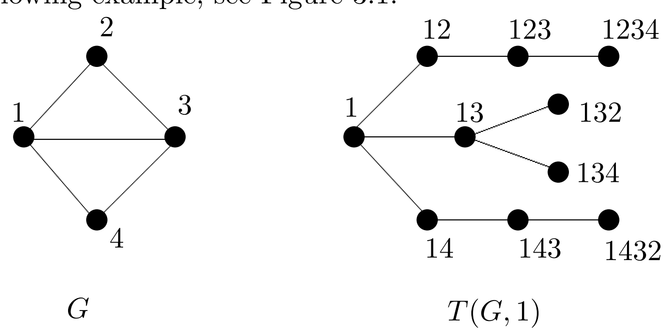


Figure 3.1

Lemma 3.2.4. *([63]) Let G be a triangle-free graph with $u \in V(G)$ and let $T = T(G,u)$ be the path tree of G which starts at u. Then*

$$\frac{h(G-u,x)}{h(G,x)} = \frac{h(T-u,x)}{h(T,x)}.$$

Theorem 3.2.4. *Let G be a graph without triangles and $u \in V(G)$ and let $T = T(G,u)$ be the path tree of G which starts at u. Then $\beta(G) = \beta(T)$.*

Proof. From Theorem 3.2.2, we have

$$\beta(G) < \beta(G-u) \text{ and } \beta(T) < \beta(T-u).$$

By Lemma 3.2.4, $\beta(G) = \beta(T)$. □

Lemma 3.2.5. *([28, 82])*

(i) For $n \geq 4$, the set of the roots of $h_1(C_n)$ is

$$\{-2(1+cos\frac{2i-1}{n}\pi) | 1 \leq i \leq \lfloor\frac{n}{2}\rfloor\};$$

(ii) For $n \geq 2$, the set of the roots of $h_1(P_n)$ is

$$\{-2(1+cos\frac{2i}{n+1}\pi) | 1 \leq i \leq \lfloor\frac{n}{2}\rfloor\}.$$

Many graphs used in the rest of this chapter have been shown in Figures 1.1, 1.2 and 1.3.

Lemma 3.2.6.
(i) $\beta(C_n) > -4$ *for* $n \geq 3$, $\beta(P_n) > -4$ *for* $n \geq 2$, $\beta(K_4^-) = -4$;
(ii) $\beta(D_n) > -4$ *for* $4 \leq n \leq 7$, $\beta(D_8) = -4$, $\beta(D_n) < -4$ *for* $n \geq 9$.

Proof: (i) From Lemma 3.2.5, we have that $\beta(C_n) > -4$, for $n \geq 3$, and $\beta(P_n) > -4$, for $n \geq 2$. Since $h_1(K_4^-) = x^2 + 5x + 4$, we have $\beta(K_4^-) = -4$.

(ii) By Theorem 1.2.4, one checks directly that $\beta(D_8) = -4$. So, by Theorem 3.2.2, (ii) is true. □

Let Z_n and V_n be two graphs with n vertices, shown in Figure 3.2.

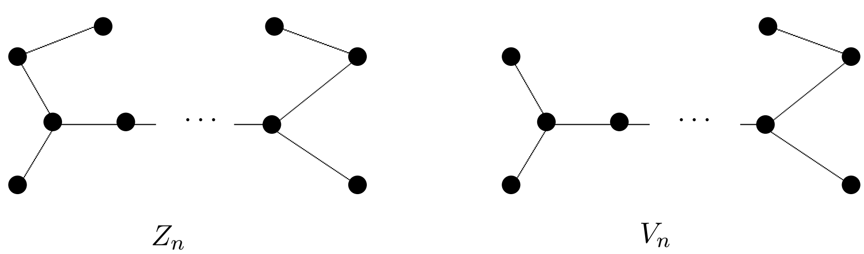


Figure 3.2

Lemma 3.2.7. *(i) For* $n \geq 4$, $h(F_{2n+1} \cup K_1) = h(B_{n+2})h(D_n)$;
(ii) For $n \geq 6$, $h(F_n \cup 2K_1) = h(Z_{n+2})$ *and* $h(B_n \cup K_1) = h(V_{n+1})$;
(iii) $h(A_7) = h(B_7) = x^7 + 7x^6 + 13x^5 + 5x^4$;
(iv) $h(A_5) = h(C_3(P_2, P_2)) = h(K_1 \cup K_4^-)$;
(v) $h(F_9 \cup K_1) = h(C_4)h(B_6)$ *and* $h(A_9 \cup K_1) = h(T_{1,1,1})h(B_6)$;
(vi) $h(B_{10}) = h(A_6)h(C_4)$ *and* $h(B_6) = h(C_3(P_2, P_3))$;
(vii) $h(F_7) = h(P_4)(x^3+5x^2+3x)$ *and* $h(F_8) = h(P_2)(x^6+8x^5+18x^4+9x^3+x^2)$.
(viii) $h(F_{11}) = h(P_2)h(A_7)(x^2 + 4x + 1)$ *and* $h(F_{17}) = h(C_3)h(K_4^-)h(B_{10})$.

Proof. (i) We can choose an edge e from F_{2n+1} such that $F_{2n+1} - e = D_n \cup D_{n+1}$. By Theorems 2.1.1 and 2.1.2,

$$h(F_{2n+1}) = h(D_n)[h(D_{n+1}) + xh(D_{n-1})].$$

Let e' be a pendant edge of B_{n+2}. From Theorem 2.1.2, we have

$$h(B_{n+2}) = xh(D_{n+1}) + x^2h(D_{n-1}).$$

So, $h(F_{2n+1} \cup K_1) = h(B_{n+2})h(D_n)$.

(ii) Note that F_n and B_n satisfy the condition of Lemma 3.2.3. So, by Lemma 3.2.3 we know that (ii) holds.

By using Theorems 2.1.1, 2.1.2 and 1.2.4, we can directly verify the equalities (iii) to (viii). □

Theorem 3.2.5. *(i)* $\beta(A_{n+1}) < \beta(A_n) < -4$, $\beta(B_n) < \beta(B_{n+1})$ *and* $\beta(F_n) < \beta(F_{n+1})$, *for all* $n \geq 6$;

(ii) $\beta(F_n) < \beta(B_n) < -4$, $\beta(F_n) < \beta(D_m)$ *and* $\beta(B_n) < \beta(D_m)$, *for all* $n \geq 6$ *and* $m \geq 4$;

(iii) $\beta(F_n) = \beta(B_m)$ *if and only if* $n = 2k + 1$ *and* $m = k + 2$, *where* $k \geq 4$;

(iv) $\beta(B_6) = \beta(A_9) < \beta(A_8) < \beta(B_7) = \beta(A_7) < \beta(B_8) < \beta(B_9) < \beta(B_{10}) = \beta(A_6)$, *and* $\beta(A_n) \leq \beta(B_m)$, *for all* $n \geq 7$ *and* $m \geq 7$, *where the equality holds if and only if* $n = m = 7$;

(v) $\beta(A_6) = \beta(F_{17})$, $\beta(A_7) = \beta(F_{11})$, $\beta(A_9) = \beta(F_9)$, *and* $\beta(A_n) \leq \beta(F_m)$, *for all* $n \geq 9$ *and* $m \geq 9$, *where the equality holds if and only if* $n = m = 9$.

Proof. (i) From Lemma 3.2.6, $\beta(C_4) > -4$. Note that $\beta(A_5) = -4$ and $h(A_6) = x(h(A_5) + h(C_4))$. By Lemma 3.2.1, it is obvious that $\beta(A_6) < \beta(A_5) = -4$. By Theorem 2.2.1, it follows that

$$h(A_n) = x(h(A_{n-1}) + h(A_{n-2})), \text{for } n \geq 7.$$

When $n = 7$, again by Lemma 3.2.1 we have that $\beta(A_7) < \beta(A_6)$. Repeating this procedure for n, $n \geq 8$, we have that $\beta(A_{n+1}) < \beta(A_n) < -4$, for $n \geq 6$.

Note that $\rho(T) > 0$ for any tree T. By Theorems 3.2.1 and 3.2.3, we have that $\rho(V_{n+2}) < \rho(V_{n+1})$ and $\beta(V_n) = -\rho^2(V_n)$. Thus $\beta(V_{n+1}) < \beta(V_{n+2})$. By Lemma 3.2.7(ii), $\beta(B_n) < \beta(B_{n+1})$, for $n \geq 6$.

Similarly, from Theorems 3.2.1 and 3.2.3 and Lemma 3.2.7(ii), one shows that $\beta(F_n) < \beta(F_{n+1})$, for $n \geq 6$.

(ii) From Theorem 2.1.2, we can get that

$$h(B_n) = x(h(D_{n-1}) + xh(D_{n-3})).$$

Hence, by Theorem 3.2.2 and Lemma 3.2.1, it is not difficult to see that $\beta(B_n) < \beta(D_{n-1})$. Since $\beta(B_n) < \beta(D_{n-1}) < \beta(D_{n-2})$, we have $\beta(B_n) < \beta(D_m)$, for all $m \le n-1$. By (i), it follows that $\beta(B_n) < \beta(B_{m+1}) < \beta(D_m)$, for all $m \ge n$.

Similarly, we can show that, for any $n \ge 6$ and $m \ge 4$, $\beta(F_n) < \beta(D_m)$.

By Lemma 3.2.6, $\beta(D_8) = -4$. Thus, by Theorem 3.2.2, we have $\beta(F_n) < \beta(B_n) < \beta(D_8) = -4$, for $n \ge 6$.

(iii) From Lemma 3.2.7, it is clear that $\beta(F_{2k+1}) = \beta(B_{k+2})$. By (i), we have that (iii) holds.

(iv) The following follows from (iii) to (v) of Lemma 3.2.7 and (i) of the theorem,

$$\beta(B_6) = \beta(A_9) < \beta(A_8) < \beta(A_7) = \beta(B_7) < \beta(B_8) < \beta(B_9) < \beta(B_{10}) = \beta(A_6).$$

So, again by (i) of the theorem, (iv) is true.

Finally, (v) can be deduced directly from Lemma 3.2.7 and the results from (i) to (iv) of the theorem. $\square$

Lemma 3.2.8. *([74]) For any $n \ge 2$, we have:*
(i) $h(T_{1,n,n+3}) = h(P_{n+1})h(A_{n+3})$,
(ii) $h(T_{1,n,n}) = h(P_n)h(A_{n+2})$,
(iii) $h(T_{1,n,2n+5}) = h(C_{n+2})h(T_{1,n+1,n+2})$,
(iv) $h(T_{2,2,n}) = h(P_2)h(A_{n+3})$,
(v) $h(T_{2,3,3}) = x^3h(P_3)(x^3 + 6x^2 + 8x + 2)$,
(vi) $\beta(T_{1,n,n}) = \beta(T_{1,n-1,n+2})$ and $\beta(T_{1,n,n+1}) = \beta(T_{1,n-1,2n+3})$.

Theorem 3.2.6. *(i) For $n \ge 2$ and $m \ge 6$,*

$$\beta(T_{1,2,m+1}) < \beta(T_{1,2,m}) < \beta(T_{1,2,5}) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1}).$$

(ii) For $3 \le l \le 11$, $n \ge 3$ and $m \ge l+3$,

$$\beta(T_{1,l,m+1}) < \beta(T_{1,l,m}) < \beta(T_{1,l,l+2}) < \beta(T_{1,l-1,n}) < \beta(T_{1,l-1,n-1}).$$

(iii) For $T_1 \in \{T_{1,l_1,l_2} | 3 \leq l_1 \leq 10, l_1 \leq l_2\}$ *and* $T_2 \in \{T_{1,l_1,l_2} | 1 \leq l_1 \leq l_2\}$*, we have* $\beta(T_1) = \beta(T_2)$ *and* $T_1 \ncong T_2$ *if and only if* $\beta(T_{1,n,n}) = \beta(T_{1,n-1,n+2})$ *and* $\beta(T_{1,n,n+1}) = \beta(T_{1,n-1,2n+3})$.

Proof. (i) and (ii) We denote by $A_{a,b}$ the graph $Q_{a+1,b+1,1}$, as shown in Figure 1.3. By Theorems 2.1.1 and 2.1.2, we have that

$$h(T_{1,l_1,l_2}) = xh(P_{l_1+l_2+1}) + xh(P_{l_1})h(P_{l_2})$$

and

$$h(A_{a,b}) = xh(T_{1,1,a+b+1}) + xh(P_a)h(T_{1,1,b}).$$

By calculation, we have $h(A_{1,1}) = x^7 + 6x^6 + 8x^5$. By Theorem 1.2.5(ii), one can get that $h(A_{a,b}) = x^2h(C_{a+b+3}) + x^2h(C_{b+2})h(P_a)$, for $b \geq 2$. From Theorem 1.2.4, by calculation we obtain the coefficients of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$, given in Tables 3.1 and 3.2. For each $h(G)$ in Tables 3.1 and 3.2, $h(G,x) = \sum\limits_{i=0}^{p(G)} b_i x^{p(G)-i}$, where $p(T_{1,l_1,l_2}) = l_1 + l_2 + 2$ and $p(A_{a,b}) = a + b + 5$.

Using Software Mathematica, we get the minimum real roots of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$, given in Table 3.3.

(l_1, l_2)	The coefficients of $h(T_{1,l_1,l_2})$: $b_0, b_1, b_2, b_3, \cdots$
$(2,5)$	$1, 8, 20, 17, 4$
$(3,5)$	$1, 9, 27, 31, 11$
$(4,6)$	$1, 11, 44, 78, 59, 15, 1$
$(5,7)$	$1, 13, 65, 157, 188, 102, 19$
$(6,8)$	$1, 15, 90, 276, 458, 400, 164, 24, 1$
$(7,9)$	$1, 17, 119, 443, 945, 1159, 776, 250, 29$
$(8,10)$	$1, 19, 152, 666, 1741, 2773, 2636, 1402, 365, 35, 1$
$(9,11)$	$1, 21, 189, 953, 2954, 5812, 7237, 5515, 2393, 515, 41$
$(10,12)$	$1, 23, 230, 1312, 4708, 11054, 17120, 17216, 10787, 3899, 706, 48, 1$
$(11,13)$	$1, 25, 275, 1751, 7143, 19517, 36274, 45644, 37982, 19958, 6111, 945, 55$

Table 3.1. The coefficients of $h(T_{1,l_1,l_2})$.

(a,b)	The coefficients of $h(A_{a,b})$: $b_0, b_1, b_2, b_3, \cdots$
$(1,1)$	$1, 6, 8$
$(2,7)$	$1, 13, 64, 148, 162, 75, 11$
$(3,7)$	$1, 14, 76, 201, 266, 160, 31$
$(4,9)$	$1, 17, 118, 430, 880, 1002, 589, 152, 13$
$(5,11)$	$1, 20, 169, 785, 2184, 3718, 3795, 2177, 610, 58$
$(6,13)$	$1, 23, 229, 1293, 4556, 10388, 15379, 14443, 8152, 2503, 351, 17$
$(7,15)$	$1, 26, 298, 1981, 8455, 24225, 47328, 62764, 55198, 30744, 10003, 1636, 93$
$(8,17)$	$1, 29, 376, 2876, 14421, 49819, 121296, 209304, 253878, 211718, 116689, 39840, 7574, 671, 21$
$(9,19)$	$1, 32, 463, 4005, 23075, 93380, 272734, 581647, 906015, 1020680, 814606, 445093, 157785, 33292, 3585, 136$
$(10,21)$	$1, 35, 559, 5395, 35119, 162981, 555750, 1414270, 2700775, 3860021, 4085950, 3142790, 1704795, 623400, 143448, 18620, 1140, 25$

Table 3.2. The coefficients of $h(A_{a,b})$.

(l_1,l_2)	$\beta(T_{1,l_1,l_2})$	(a,b)	$\beta(A_{a,b})$
$(2,5)$	-4.0000	$(1,1)$	-4.00000
$(3,5)$	-4.09529	$(2,7)$	-4.09529
$(4,6)$	-4.16035	$(3,7)$	-4.15875
$(5,7)$	-4.19353	$(4,9)$	-4.18970
$(6,8)$	-4.21145	$(5,11)$	-4.20829
$(7,9)$	-4.22153	$(6,13)$	-4.21937
$(8,10)$	-4.22736	$(7,15)$	-4.22597
$(9,11)$	-4.23080	$(8,17)$	-4.22993
$(10,12)$	-4.23286	$(9,19)$	-4.23232
$(11,13)$	-4.23411	$(10,21)$	-4.23378

Table 3.3. The minimum real roots of $h(T_{1,l_1,l_2})$ and $h(A_{a,b})$.

By Theorems 3.2.2 and 3.2.3, we have

$$\beta(A_{a,b}) < \beta(A_{a,b+1}) < \beta(A_{a,b+2}) < \cdots < \beta(A_{a,b+k}), \text{for } k \geq 3, \tag{3.1}$$

and

$$\beta(A_{a,b}) < \beta(T_{1,a,b+2}). \tag{3.2}$$

From Table 3.3, one sees that $\beta(T_{1,2,5}) = \beta(A_{1,1})$ and $\beta(T_{1,3,5}) = \beta(A_{2,7})$, and $\beta(T_{1,l+1,l+3}) < \beta(A_{l,2l+1})$ for $3 \le l \le 10$. So, by (3.1) and (3.2) and Theorem 3.2.3, we have:

(a) for $l = 1$, $m \ge 6$ and $n \ge 2$, $\beta(T_{1,2,m+1}) < \beta(T_{1,2,m}) < \beta(T_{1,2,5}) = \beta(A_{1,1}) = \beta(U_n) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1})$,

(b) for $l = 2$, $m \ge 6$ and $n \ge 2$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) = \beta(A_{2,7}) < \beta(A_{2,n+6}) < \beta(T_{1,2,n}) < \beta(T_{1,2,n-1})$,

(c) for $3 \le l \le 10$, $m \ge l+4$ and $n \ge 2$, $\beta(T_{1,l+1,m+1}) < \beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+3}) < \beta(A_{l,2l+1}) < \beta(A_{l,n+2l}) < \beta(T_{1,l,n}) < \beta(T_{1,l,n-1})$.

Thus, from (a), (b) and (c), we know that (i) and (ii) of the theorem hold.

(iii) By (i) and (ii) of the theorem and Lemma 3.2.8(ii), we have:

(d) for $m \ge 6$ and $n \ge 2$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) < \beta(T_{1,3,4}) < \beta(T_{1,3,3}) = \beta(T_{1,2,5}) < \beta(T_{1,1,n}) < \beta(T_{1,1,n-1})$,

(e) for $m \ge 6$ and $n \ge 12$, $\beta(T_{1,3,m+1}) < \beta(T_{1,3,m}) < \beta(T_{1,3,5}) < \beta(T_{1,2,n}) < \beta(T_{1,2,n-1}) < \beta(T_{1,2,10}) < \beta(T_{1,2,9}) = \beta(T_{1,3,4}) < \beta(T_{1,2,8}) < \beta(T_{1,2,7}) < \beta(T_{1,2,6}) < \beta(T_{1,3,3}) = \beta(T_{1,2,5}) < \beta(T_{1,2,4}) < \beta(T_{1,2,3}) < \beta(T_{1,2,2})$,

(f) for $3 \le l \le 10$, $m \ge l+4$ and $n \ge 2l+8$, $\beta(T_{1,l+1,m+1}) < \beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+3}) < \beta(T_{1,l,n}) < \beta(T_{1,l,n-1}) < \beta(T_{1,l,2l+6}) < \beta(T_{1,l+1,l+2}) = \beta(T_{1,l,2l+5}) < \beta(T_{1,l,2l+4}) < \cdots < \beta(T_{1,l,l+5}) < \beta(T_{1,l,l+4}) < \beta(T_{1,l+1,l+1}) = \beta(T_{1,l,l+3}) < \beta(T_{1,l,l+2}) < \beta(T_{1,l,l+1}) < \beta(T_{1,l,l})$,

(g) for $l \ge 11$, $m \ge l+1$ and $n \ge 2$, $\beta(T_{1,l+1,m}) < \beta(T_{1,l+1,l+1}) \le \beta(T_{1,12,12}) = \beta(T_{1,11,13}) < \beta(T_{1,10,n})$, by Lemma 3.2.8(vi).

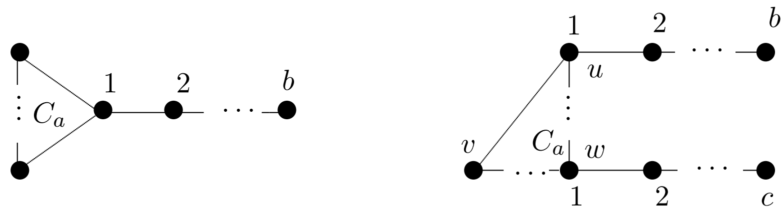
From (d), (e), (f) and (g), it is not difficult to see that (iii) holds. $\square$

From the theorem, we propose the following.

Problem 3.2.1. *Is it true that* $\beta(T_{1,l,l+2}) < \beta(T_{1,l-1,n})$*, for all* $l \ge 3$ *and* $n \ge 1$*?*

3.3 Graphs G with $\beta(G) \in [-4, 0]$

In this section, we determine all connected graphs with $\beta(G) \geq -4$. For $a \geq 4$ and $b \geq 2$, we denote by $C_a(P_b)$ the graph obtained from C_a and P_b by identifying a vertex of C_a with an end-vertex of P_b. For $a \geq 3$, $b \geq 2$ and $c \geq 2$, let u and w be two different vertices of C_a, we denote by $C_a(P_b, P_c)$ the graph obtained from C_a, P_b and P_c by identifying u with an end-vertex of P_b and w with an end-vertex of P_c, respectively, see Figure 3.3.



$C_a(P_b)$ $\qquad$ $C_a(P_b, P_c)$

Figure 3.3

Lemma 3.3.1. *(i) $\beta(C_n(P_m)) \leq -4$, for $n \geq 4$ and $m \geq 2$, and the equality holds if and only if $n = 4$ and $m = 2$;*
(ii) $\beta(C_n(P_{m_1}, P_{m_2})) \leq -4$, for $n \geq 3$ and $m_i \geq 2$, where $i = 1, 2$, and the equality holds if and only if $n = 3$ and $m_1 = m_2 = 2$.

Proof. (i) We prove (i) by considering the following cases.

Case 1. $n = 4$ and $m = 2$.

Since $h_1(C_4(P_2)) = x^2 + 5x + 4$, we have $\beta(C_4(P_2)) = -4$.

Case 2. $n = 4$ and $m \geq 3$.

Clearly, $C_4(P_2)$ is a proper subgraph of $C_4(P_m)$. We have $\beta(C_4(P_m)) < -4$, by Theorem 3.2.2.

Case 3. $n \geq 5$ and $m = 2$.

Note that $h_1(C_5(P_2)) = x^3+6x^2+8x+1$. By calculation, $\beta(C_5(P_2)) < -4$. When $n \geq 6$, it follows, from Theorem 2.2.1, that

$$h(C_n(P_2)) = x(h(C_{n-1}(P_2)) + h(C_{n-2}(P_2)).$$

Since $\beta(C_5(P_2)) < \beta(C_4(P_2)) = -4$ and $\partial h(C_n(P_2)) = \partial(xh(C_{n-2}(P_2))) - 1$, we know, from Lemma 3.2.1, that

$$\beta(C_n(P_2)) < \beta(C_{n-1}(P_2)) < \cdots < \beta(C_4(P_2)) = -4.$$

Case 4. $n \geq 5$ and $m \geq 3$.

Since $C_n(P_2)$ is a proper subgraph of $C_n(P_m)$, by Theorem 3.2.2 we have $\beta(C_n(P_m)) < -4$.

From Case 1 to Case 4, we have that, for $n \geq 4$ and $m \geq 2$, $\beta(C_n(P_m)) < -4$ except for $\beta(C_4(P_2)) = -4$. This completes the proof of (i).

(ii) We distinguish the following cases.

Case 1. $n = 3$ and $m_1 = m_2 = 2$.

By calculation, we have that $h_1(C_3(P_2, P_2)) = x^2 + 5x + 4$. So, we get easily that $\beta(C_3(P_2, P_2)) = -4$.

Case 2. $n = 3$ and $m_1 \geq 3$, or $m_2 \geq 3$.

It is not hard to see that $C_3(P_2, P_2)$ is a proper subgraph of $C_3(P_{m_1}, P_{m_2})$, for $n = 3$ and $m_1 \geq 3$, or $n = 3$ and $m_2 \geq 3$. It follows, from Theorem 3.2.2, that $\beta(C_3(P_{m_1}, P_{m_2})) < -4$.

Case 3. $n \geq 4$, $m_1 \geq 2$ and $m_2 \geq 2$.

Obviously, $C_n(P_{m_1}, P_{m_2})$ must have a proper subgraph $C_n(P_{m_1})$, where $m_1 \geq 2$ and $n \geq 4$. By Theorem 3.2.2 and (i), we have $\beta(C_n(P_{m_1}, P_{m_2})) < -4$.

From the above cases, it is not difficult to see that, for $n \geq 4$, $m_1 \geq 2$ and $m_2 \geq 2$, $\beta(C_n(P_{m_1}, P_{m_2})) < -4$ except for $\beta(C_3(P_2, P_2)) = -4$. $\square$

Lemma 3.3.2. *([19, 21]) Let T be a tree. Then*
(i) $\rho(G) = 2$ if and only if

$$G \in \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}\} \cup \mathcal{U}.$$

(ii) $\rho(G) < 2$ if and only if $G \in \{K_1, T_{1,2,i} | i = 2, 3, 4\} \cup \mathcal{P} \cup \mathcal{T}_1$.

Theorem 3.3.1. *Let G be a connected graph without triangles. Then*
(i) $\beta(G) = -4$ if and only if

$$G \in \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2)\} \cup \mathcal{U};$$

(ii) $\beta(G) > -4$ if and only if $G \in \{K_1, T_{1,2,i} | i = 2, 3, 4\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1$.

Proof. We prove the theorem by distinguishing the following cases.

Case 1. G is a tree.

By Theorem 3.2.1, $\beta(G) = -(\rho(G))^2$. The theorem follows from Lemma 3.3.2 immediately.

Case 2. G is a connected graph without triangles and $q(G) \geq p(G)$.

If $p(G) \geq 6$ and $G \ncong C_n$, then G must contain either a subgraph $C_n(P_2)$, where $n \geq 5$, or a proper subgraph $C_4(P_2)$. By Theorem 3.2.2 and Lemma 3.3.1, we have that $\beta(G) < \beta(C_4(P_2)) = -4$, or $\beta(G) \leq \beta(C_n(P_2)) < -4$, for $n \geq 5$. If $p(G) \leq 5$ or $G \cong C_n$, then G must be C_n or $C_4(P_2)$. By Lemmas 3.2.6 and 3.3.1, $\beta(C_n) > -4$, for $n \geq 3$, and $\beta(C_4(P_2)) = -4$. Hence if G is a connected graph without triangles and $q(G) \geq p(G)$, then $\beta(G) = -4$ if and only if $G = C_4(P_2)$ and $\beta(G) > -4$ if and only if $G = C_n$, where $n \geq 3$.

This completes the proof of the theorem. □

Theorem 3.3.2. *Let G be a connected graph. Then*
(i) $\beta(G) = -4$ if and only if

$$G \in \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2, P_2), K_4^-, D_8\} \cup \mathcal{U};$$

(ii) $\beta(G) > -4$ if and only if

$$G \in \{K_1\} \cup \{T_{1,2,i} | i = 2, 3, 4\} \cup \{D_i | i = 4, 5, 6, 7\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1.$$

Proof. Suppose that G is a graph without triangles. Then, from Theorem 3.3.1, the theorem holds.

Suppose that G contains only one triangle. It is not hard to see that any graph G, except for D_i, $4 \leq i \leq 8$, $C_3(P_2, P_2)$ and C_3, contains a proper subgraph G^* such that $G^* \in \{D_8, C_3(P_2, P_2),\ K_{1,4}, U_n | n \geq 6\}$. By Lemma 3.2.6, we have that $\beta(D_8) = -4$, $\beta(D_i) > -4$, for $i = 4, 5, 6, 7$, and $\beta(C_3) > -4$. Note that $\beta(C_3(P_2, P_2)) = \beta(K_{1,4}) = \beta(U_n) = -4$. Hence the theorem follows from Theorem 3.2.2.

Suppose that G contains at least two triangles. Then, any graph, except for K_4^-, must contain a proper subgraph G^* such that $G^* \in \{U_n,\ C_3(P_2, P_2),\ K_4^-, K_{1,4} | n \geq 6\}$. Since $\beta(G^*) = \beta(K_4^-) = -4$, by Theorem 3.2.2 we have that the theorem holds. □

From Theorem 3.3.2, we have the following corollaries.

Corollary 3.3.1. *Let G be a connected graph. Then $\beta(G) \geq -3$ if and only if $G \in \{P_2, P_3, P_4, P_5, C_3, T_{1,1,1}, K_1\}$.*

Theorem 3.3.2 means that the minimum real roots of $\sigma(G, x)$ are greater than or equal to -4 if and only if each component of $\overline{G}$ is one of subgraphs of the following graphs:

$$T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2, P_2), K_4^-, D_8, U_{n+3}, C_n, \text{where } n \geq 3.$$

Corollary 3.3.2. *Let G be a connected graph with $\beta(G) \geq -4$. Then all the roots of $\sigma(\overline{G}, x)$ are real.*

3.4 Graphs G with $\beta(G) \in [-(2+\sqrt{5}), -4)$

In this section, our goal is to determine all connected graphs with $\beta(G) \in [-(2+\sqrt{5}), -4)$. Let $T_{a,b,c}$ and $Q_{a,b,c}$ be the graphs shown in Figure 1.3. The following lemmas can be found from [19, 20].

Lemma 3.4.1. *([19]) If G is a tree, then $2 < \rho(G) \leq (2+\sqrt{5})^{1/2}$ if and only if G is one of the following graphs:*
(i) $T_{i,j,k}$ for $i = 1, j = 2, k > 5$, or $i = 1, j > 2, k > 3$, or $i = j = 2, k > 2$, or $i = 2, j = k = 3$.
(ii) $Q_{i,j,k}$ for $(i, j, k) \in \{(2, 1, 2), (3, 4, 2), (3, 5, 3), (4, 7, 3), (4, 8, 4)\}$, or $i \geq 2, j \geq j^*(i, k), k \geq 1$ where $(i, k) \neq (2, 1)$ and

$$j^*(i,k) = \begin{cases} i+k+1 & for \quad i \geq 4, \\ 3+k & for \quad i = 3, \\ k & for \quad i = 2. \end{cases}$$

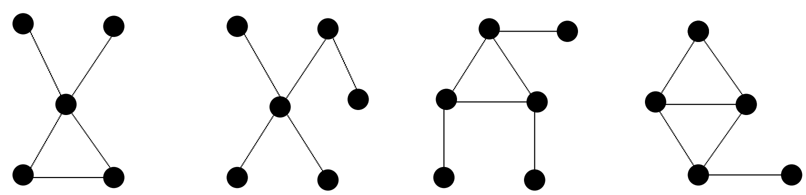
From Theorem 3.2.1 and Lemma 3.4.1, we have

Lemma 3.4.2. *If G is a tree, then $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the graphs listed by Lemma 3.4.1.*

Lemma 3.4.3. *([20]) Let $G \ncong K_{1,4}$. If G is a connected graph with at least three vertices of degree 3 or with at least one vertex of degree 4, then $\rho(G) > (2+\sqrt{5})^{1/2}$.*

Lemma 3.4.4. *Let G be a connected graph with at least three vertices of degree 3 or with at least one vertex of degree 4. If G is not a tree, then* $\beta(G) < -(2+\sqrt{5})$.

Proof. Suppose that G has at least one vertex of degree 4 and G is not a tree. Then G must contain a subgraph Q_i, as shown in Figure 3.4, where $i = 1, 2$.



Q_1 Q_2 Q_3 Q_4

Figure 3.4

By calculation, we have that $h_1(Q_1) = h_1(Q_2) = x^2+5x+3$ and $h_1(K_4) = h_1(Q_3) = h_1(Q_4) = x^3+6x^2+7x+1$. It is not hard to verify that $\beta(Q_i) < -(2+\sqrt{5})$, for each i. From Theorem 3.2.2, the lemma follows.

Suppose that G has at least three vertices of degree 3 and G is not a tree. We distinguish the following three cases:

Case 1. G is a triangle-free graph.

Let u, v and w be three vertices of degree 3 in G and let $N_G(u) = \{v_1, w_1, u'\}$, $N_G(v) = \{v_i, v', v''\}$ and $N_G(w) = \{w_j, w', w''\}$. Let $T = T(G,u)$ be the path tree of G which starts at u. Since G is connected, we assume that $uv_1v_2\cdots v_iv$ (respectively $uw_1w_2\cdots w_jw$) is a shortest path between u and v (respectively u and w). Clearly, $v' \neq v_i$. One sees that $v' \neq u$ and $v' \neq v_t$, for $t = 1, 2, \cdots, i-1$ (otherwise $uv_1\cdots v_tv$ is a shorter path than $uv_1v_2\cdots v_iv$, which is a contradiction.). Thus, we have that $v', v'' \notin \{u\} \cup \{v_t | t = 1, 2, \cdots, i\}$ and $w', w'' \notin \{u\} \cup \{w_t | t = 1, 2, \cdots, j\}$. It is not hard to see that (u), $(uv_1v_2\cdots v_iv)$ and $(uw_1w_2\cdots w_jw)$ are three different vertices of degree 3 in the path tree T of G. From Lemma 3.4.3 and Theorem 3.2.4, it follows that $\beta(G) = \beta(T) < -(2+\sqrt{5})$.

Case 2. G contains three vertices of degree 3, say v_1, v_2, v_3, such that v_1, v_2 and v_3 are vertices of the triangle $v_1v_2v_3$.

It is not difficult to see that G must contain a subgraph H such that $H \in \{K_4, Q_3, Q_4\}$. Since $\beta(H) < -(2+\sqrt{5})$, by Theorem 3.2.2 we have $\beta(G) = \beta(H) < -(2+\sqrt{5})$.

Case 3. G has triangles and each triangle of G contains at least one vertex of degree 2. By splitting a vertex of degree 2 of each triangle in G until the graph does not contain triangle any more, we obtain a graph H. By Lemma 3.2.3, it follows that $h_1(H) = h_1(G)$. Clearly, H contains at least three vertices of degree 3 and has no triangle. So, by Case 1, $\beta(G) = \beta(H) < -(2+\sqrt{5})$. $\square$

Lemma 3.4.5. *Let G be a connected graph with only one vertex of degree 3 and with at least one cycle. Then $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the following graphs: $G \cong C_3(P_b)$, for $b \geq 7$, or $G \cong C_a(P_2)$, for $a \geq 5$, or $G \cong C_4(P_3)$, where $C_a(P_b)$ is the graph shown in Figure 3.3.*

Proof. Clearly, G is connected and has only one vertex of degree 3 and at least one cycle if and only if $G \cong C_a(P_b)$, where $a \geq 3$ and $b \geq 2$.

Suppose $a = 3$. From Theorem 1.2.5, $h_1(C_a(P_b)) = h_1(T_{1,2,b-1})$. From Lemma 3.4.2, $\beta(T_{1,2,b-1}) \in [-(2+\sqrt{5}), -4)$ if and only if $b \geq 7$.

Suppose $a \geq 4$. Then $C_a(P_b)$ does not contain a triangle. Let $v \in V(G)$ and $d(v) = 3$ and let $T = T(G, v)$. From Theorem 3.2.4, $\beta(G) = \beta(T)$ and $T = T_{a-1,a-1,b-1}$. From Lemma 3.4.2, $-(2+\sqrt{5}) \leq \beta(T) < -4$ if and only if $a = 4$ and $b = 3$ or $b = 2$ and $a \geq 5$. So, from Theorem 3.2.4, we have that $-(2+\sqrt{5}) \leq \beta(C_a(P_b)) < -4$ if and only if $a = 4$ and $b = 3$ or $b = 2$ and $a \geq 5$.

From the above arguments, the lemma follows. $\square$

In the rest of this section, we use the following graphs shown in Figure 3.5.

Lemma 3.4.6. *Let G be a connected graph with only two vertices of degree 3 and at least one cycle. Then, $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the following graphs:*
(i) $C_3(P_2, P_3)$;

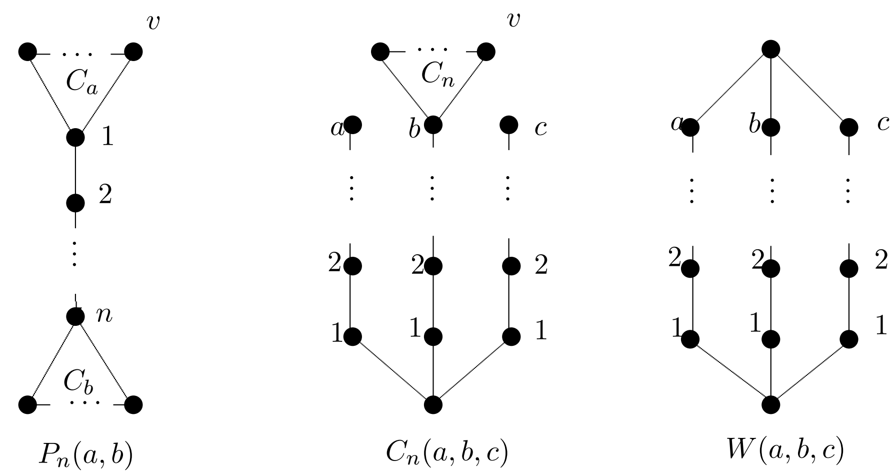


Figure 3.5

(ii) $P_n(a,b)$ *for* $a = b = 3$ *and* $n \geq 5$.

(iii) $C_n(a,b,c)$ *for* $a = 1, n = 3$ *and* $b = 5$ *and* $c = 3$*, or* $b \geq 1$ *if* $c = 1$*, or* $b \geq 4$ *if* $c = 2$*, or* $b \geq c + 3$ *if* $c \geq 3$.

Proof. Let G be a connected graph with only two vertices of degree 3 and at least one cycle. Then G is one of the following graphs.

$$C_a(P_b, P_c), P_n(a,b), C_n(a,b,c), W(a,b,c),$$

where $C_a(P_b, P_c)$ is the graph shown in Figure 3.3 and the others are the graphs shown in Figure 3.5.

We distinguish the following cases:

Case 1. G has no triangle and $G \in \{C_a(P_b, P_c), P_n(a,b), C_n(a,b,c), W(a,b,c)\}$.

Obviously, G must contain at least one cycle of length at least 4. Now we choose a vertex x in G such that the path tree $T = T(G,x)$ which starts at x contains at least three vertices of degree 3. For $P_n(a,b)$ and $C_n(a,b,c)$, take $x = v$, shown in Figure 3.5. For $C_a(P_b, P_c)$, take $x = v$ if uw is an edge of $C_a(P_b, P_c)$, otherwise $x = u$, shown in Figure 3.3. For $W(a,b,c)$, we choose a vertex of degree 3 as x if $min\{a,b,c\} \geq 1$, otherwise x is a vertex of degree 2 in $W(a,b,c)$. Recalling that every G has no triangle, we see that each $T = T(G,x)$ contains at least three vertices of degree 3. Thus, by Theorem

3.2.4 and Lemma 3.4.4, we have $\beta(G) = \beta(T) < -(2+\sqrt{5})$.

Case 2. G has some triangles.

Case 2.1. $C_a(P_b, P_c)$ and $a = 3$.

Splitting the vertex of degree 2 in C_3 of $C_a(P_b, P_c)$, from Lemma 3.2.3 we have $h_1(C_a(P_b, P_c)) = h_1(Q_{b,1,c-1})$. By Lemma 3.4.2, there is only one graph $Q_{2,1,2}$ such that $\beta(G) = \beta(Q_{2,1,2}) > -(2+\sqrt{5})$. So, $b = 2$ and $c = 3$. Hence $\beta(C_3(P_2, P_3)) > -(2+\sqrt{5})$, which is (i).

Case 2.2. $C_n(a, b, c)$ and $n = 3$.

Let H be the graph obtained from $C_n(a, b, c)$ by splitting the vertex v of degree 2 in C_3. By Lemmas 3.2.3 and 3.4.2, we have that if $min\{a, c\} \geq 2$, then $\beta(G) = \beta(H) < -(2+\sqrt{5})$. So, $min\{a, c\} = 1$. Assume that $a = 1$. Then $H \cong Q_{3,b,c}$ and $\beta(C_3(1, b, c)) = \beta(Q_{3,b,c})$. Note that $Q_{3,b,c} \cong Q_{c+1,b,2}$. By Lemma 3.4.2, it follows that $\beta(Q_{3,b,c}) > -(2+\sqrt{5})$ if and only if $b = 5$ and $c = 3$, or $b \geq 1$ for $c = 1$, or $b \geq 4$ for $c = 2$, or $b \geq c+3$ for $c \geq 3$, which implies that (iii) holds.

Case 2.3. $P_n(a, b))$ and $a = 3$ or $b = 3$.

Suppose that $P_n(a, b)$ has exactly one triangle, say $a = 3$ and $b \geq 4$. Let H denote the graph obtained by splitting the vertex v of degree 2 in C_3 of $P_n(a, b)$. By Lemma 3.2.3, $h_1(H) = h_1(P_n(a, b))$ and $H \in \{C_b(1, n-1, 2) | n \geq 4, b \geq 4\}$. From the proof of Case 1, we have $\beta(H) < -(2+\sqrt{5})$ for each H.

Suppose that $P_n(a, b)$ has exactly two triangles. Clearly, $a = b = 3$. Splitting a vertex of degree 2 in each C_3 of $P_n(a, b)$, we get the graph $Q_{3,n-1,2}$. From Lemma 3.2.3, $h_1(P_n(3, 3)) = h_1(Q_{3,n-1,2})$. By Lemma 3.4.2, $\beta(Q_{3,n-1,2}) > -(2+\sqrt{5})$ if and only if $n \geq 5$, which is (ii).

Case 2.4. $W(a, b, c)$.

Suppose that $W(a, b, c)$ contains only one triangle. Assume that $a \geq 2$, $b = 1$ and $c = 0$. Splitting the vertex of degree 2 in C_3 of $W(a, b, c)$, by Lemma 3.2.3 we get a graph H without triangles such that $h_1(H) = h_1(G)$ and $H \in \{C_{a+2}(P_2, P_2)\}$. By the proof of case 1, there is no graph H such that $\beta(H) > -(2+\sqrt{5})$.

Suppose that $P_n(a, b)$ contains exactly two triangles. Then $W(a, b, c) \cong K_4^-$. It can be verified that $\beta(K_4^-) = -4$.

From the above arguments, we know that the theorem holds. □

From Lemmas 3.4.2, 3.4.4, 3.4.5 and 3.4.6, we find all connected graphs with $-(2+\sqrt{5}) \leq \beta(G) < -4$. Note that $C_3(P_b) = D_{b+2}$, for $b \geq 2$, $C_a(P_2) = A_{a+1}$, for $a \geq 4$, and $P_n(3,3) = F_{n+4}$, for $n \geq 2$. So, we have

Theorem 3.4.1. *Let G be a connected graph. Then $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the following graphs:*

(i) $T_{a,b,c}$ for $a=1, b=2, c>5$, or $a=1, b>2, c>3$, or $a=b=2, c>2$, or $a=2, b=c=3$;

(ii) $Q_{a,b,c}$ for $(a,b,c) \in \{(2,1,2),(3,4,2),(3,5,3),(4,7,3),(4,8,4)\}$, or $a \geq 2, b \geq b^*(a,c), c \geq 1$, where $(a,c) \neq (2,1)$ and

$$b^*(a,c) = \begin{cases} a+c+1 & for \quad a \geq 4, \\ 3+c & for \quad a = 3, \\ c & for \quad a = 2; \end{cases}$$

(iii) D_n for $n \geq 9$;

(iv) A_n for $n \geq 6$;

(v) F_n for $n \geq 9$;

(vi) $C_3(a,b,c)$ for $a=1, b=5$ and $c=3$, or $a=1$ and $b \geq 1$ if $c=1$, or $a=1$ and $b \geq 4$ if $c=2$, or $a=1$ and $b \geq c+3$ if $c \geq 3$;

(vii) $G \cong C_4(P_3)$, or $G \cong C_3(P_2,P_3)$.

From (1.1) and Theorem 3.3.2, we have

Theorem 3.4.2. *Let G be a connected graph with the minimum real roots of its σ-polynomial in the interval $[-(2+\sqrt{5}),-4)$. Then, every component of $\overline{G}$ is one of the graphs listed in Theorem 3.4.1.*

Corollary 3.4.1. *Let G be a connected graph with $\beta(G) \geq -(2+\sqrt{5})$. Then all the roots of $\sigma(\overline{G},x)$ are real.*

3.5 The complex roots of adjoint polynomials of graphs

For a graph G with n vertices, if $\sigma(G,x)$ has at least one complex root, then G is said to be a *σ-unreal graph.* We define $\eta(G) = |E(G)|/\binom{n}{2}$, where $\eta(G)$

is said to be *the edge-density* of G. We denote by $\eta(n)$ the minimum edge-density over all σ-unreal graphs with n vertices. In [5], Brenti, Royle and Wagner determined all σ-unreal graphs with 8 and 9 vertices. Furthermore, they proposed the following problem.

Problem 3.5.1. *([5]) For a positive integer n, let $\eta(n)$ be the minimum edge-density over all σ-unreal graphs with n vertices. Give a good lower bound for $\eta(n)$. In particular, is there a constant $c > 0$ such that $\eta(n) > c$ for sufficiency large n?*

In this section, we study the unreal roots of $\sigma(G, x)$ by applying the results of adjoint polynomials. We establish a way of constructing σ-unreal graphs and give a negative answer to Problem 3.5.1.

If all roots of $h(G, x)$ are real roots, then G is called *h-real*, otherwise G is called *h-unreal*. From (1.1), we have

Lemma 3.5.1. *For any graph G, G is h-unreal if and only if $\overline{G}$ is σ-unreal.*

Let H and G be two graphs and let $v \in V(H)$ and $u \in V(G)$. By $G_u^t(H_v)$ we denote the graph obtained from G and t copies of H and a star $K_{1,t}$ by identifying every vertex of degree 1 of $K_{1,t}$ with vertex v of a copy of H and identifying the center of $K_{1,t}$ with vertex u of G, see Figure 1.4.

Lemma 3.5.2. *Let H and G be two graphs and let $v \in V(H)$ and $u \in V(G)$. Then*

$$h(G_u^t(H_v), x) = [h(H, x)]^t \left[h(G, x) + \frac{txh(H - v, x)}{h(H, x)} h(G - u, x) \right].$$

Proof. By induction on t. When $t = 1$, by Theorems 2.1.1 and 2.1.2 we have

$$\begin{aligned} h(G_u(H_v), x) &= h(H, x)h(G, x) + xh(H - v, x)h(G - u, x) \\ &= [h(H, x)][h(G, x) + \tfrac{xh(H-v,x)}{h(H,x)} h(G - u, x)]. \end{aligned}$$

Suppose that the result holds for k, $k \geq 2$. By Theorems 2.1.1 and 2.1.2,

$$h(G_u^{k+1}(H_v), x) = h(G_u^k(H_v), x)h(H, x) + xh(H - v, x)[h(H, x)]^k h(G - u, x).$$

By the induction hypothesis, we have

$$\begin{aligned} h(G_u^{k+1}(H_v), x) &= [h(H,x)]^{k+1}h(G,x) + xk[h(H,x)]^k h(H-v,x)\times \\ &\qquad h(G-u,x) + xh(H-v,x)[h(H,x)]^k h(G-u,x) \\ &= [h(H,x)]^{k+1}\left[h(G,x) + \tfrac{(k+1)xh(H-v,x)}{h(H,x)}h(G-u,x)\right]. \end{aligned}$$

□

The following two theorems follow directly from Theorems 2.1.1 and 2.1.2, and Lemmas 3.5.1 and 3.5.2.

Theorem 3.5.1. *Let $\overline{H}$ be a σ-unreal graph and let G be a graph with k components G_1, G_2, ..., G_k. Then $\overline{H \cup (\cup_{i=1}^{k} G_i)}$ is σ-unreal.*

Theorem 3.5.2. *Let $\overline{H}$ be a σ-unreal graph and let G be an arbitrary graph, and let $v \in V(H)$ and $u \in V(G)$. If $t \geq 2$, then $\overline{G_u^t(H_v)}$ is σ-unreal.*

We now construct two classes of σ-unreal graphs such that $\eta(G) \to 0$ as $n \to \infty$.

Class 1. Let $\overline{H}$ be a σ-unreal graph with m vertices, where m is a constant. By Theorem 3.5.1, $\overline{K_{n-m} \cup H}$ is σ-unreal. Since

$$(n-m)m < |E(\overline{K_{n-m} \cup H})| < (n-m)m + \binom{m}{2}$$

and $V(\overline{K_{n-m} \cup H}) = n$, we have $\eta(\overline{K_{n-m} \cup H}) \to 0$ as $n \to \infty$.

Class 2. Let $\overline{H}$ be a σ-unreal graph with m vertices and $G = K_{n-tm}$, and let $v \in V(H)$ and $u \in V(G)$, where m and t are constants. By Theorem 3.5.2, $\overline{G_u^t(H_v)}$, for $t \geq 2$, is σ-unreal. Since

$$t(n-tm)m + \binom{t}{2}m^2 - t < |E(\overline{G_u^t(H_v)})| < t(n-tm)m + \binom{t}{2}m^2 + \binom{m}{2}t - t$$

and $V(\overline{G_u^t(H_v)}) = n$, we have $\eta(\overline{G_u^t(H_v)}) \to 0$ as $n \to \infty$.

In [5], all σ-unreal graphs with 8 and 9 vertices, namely, 2 σ-unreal graphs with 8 vertices and 22 σ-unreal graphs with 9 vertices, were given. Without loss of generality, assume that H is a graph with m vertices such that $\overline{H}$ is a σ-unreal graph. Since n is arbitrarily large, we may assume $n \geq tm+1$ for any positive integer t. Take $H_i = K_{n-im}$, where $i = 1, 2, \ldots, t$. Let $v \in V(H)$. By $H_i^i(H_v)$ we denote the graph obtained from H_i and i copies of H and a star $K_{1,i}$ by identifying every vertex of degree 1 of $K_{1,i}$ with vertex v of a copy of H and identifying the center of $K_{1,i}$ with a vertex of H_i, where $i = 2, 3, \ldots, t$. Note that $|V(H_1 \cup H)| = |V(H_i^i(H_v))| = n$. From the above discussion, it is not difficult to see that

(i) $\overline{H_1 \cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}$, $\ldots$, $\overline{H_t^t(H_v)}$ is a σ-unreal graph sequence, i.e., each graph of the graph sequence is σ-unreal;

(ii) $\eta(\overline{H_1 \cup H}) \to 0$ and $\eta(\overline{H_i^i(H_v)}) \to 0$ as $n \to \infty$, where $i = 2, 3, \ldots, t$. Therefore, from the definition of $\eta(n)$, it is clear that $\eta(n) \to 0$ as $n \to \infty$. So, we have the following result.

Theorem 3.5.3. *Let H be a graph with m vertices and $v \in V(H)$ such that $\overline{H}$ is σ-unreal. Let t be a positive integer and $H_i = K_{n-mi}$. Then there exists a σ-unreal graph sequence $\overline{H_1 \cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}$, $\ldots$, $\overline{H_t^t(H_v)}$ such that $\eta(\overline{H_1 \cup H}) \to 0$ and $\eta(\overline{H_i^i(H_v)}) \to 0$ as $n \to \infty$, where $i = 2, 3, \ldots, t$, moreover $\eta(n) \to 0$ as $n \to \infty$.*

It is obvious that Theorem 3.5.3 answers Problem 3.5.1 negatively.

For any rational number $c = \frac{p}{q}$ with $0 \leq c \leq 1$, p and q are positive integers and $p \leq q$. In the following we construct two classes of graphs G such that $\eta(G) \to c$ as $n \to \infty$. Let s be a constant. Without any confusion, we simply denote by $K_n - s$ the graph obtained by removing s edges from K_n. Since $q(K_{n-tm}) > \frac{c}{2}[(n-m)^2 - s_1]$, for sufficiency large n, we may take $G = K_{n-m} - \frac{c}{2}[(n-m)^2 - s_1]$ and $F = K_{n-tm} - \frac{c}{2}[(n-tm)^2 - s_2]$, where $2q|[(n-m)^2 - s_1]$ and $2q|[(n-tm)^2 - s_2]$, $0 \leq s_1, s_2 < 2q$. Let $\overline{H}$ be a σ-unreal graph with m vertices, and let $v \in V(H)$ and $u \in V(F)$. By Theorem 2.1.1 and Lemma 3.5.1 we have that $G_1 = \overline{G \cup H}$ and $G_2 = \overline{F_t^t(H_v)}$ are σ-unreal graphs with n vertices, where $t \geq 2$. Note that

$$(n-m)m + \frac{c}{2}[(n-m)^2 - s_1] < |E(G_1)| < (n-m)m + \binom{m}{2} + \frac{c}{2}[(n-m)^2 - s_1]$$

and

$$(n-tm)m+\binom{t}{2}m^2+\frac{c}{2}[(n-tm)^2-s_2]-t<|E(G_2)|<$$

$$(n-tm)m+\binom{t}{2}m^2+\binom{m}{2}t+\frac{c}{2}[(n-tm)^2-s_2]-t.$$

Since m, s_1, s_2, c and t are constants, we have

$$\lim_{n\to\infty}\frac{E(G_1)}{\binom{n}{2}}=\lim_{n\to\infty}\frac{\frac{c}{2}[(n-m)^2-s_1]}{\frac{n(n-1)}{2}}=c$$

and

$$\lim_{n\to\infty}\frac{E(G_2)}{\binom{n}{2}}=\lim_{n\to\infty}\frac{\frac{c}{2}[(n-tm)^2-s_2]}{\frac{n(n-1)}{2}}=c.$$

Hence

$$\lim_{n\to\infty}\eta(G_1)=\lim_{n\to\infty}\eta(G_2)=c.$$

From the above argument, we have the following result.

Theorem 3.5.4. *Let H be a graph with m vertices and $v \in V(H)$ such that $\overline{H}$ is σ-unreal. Let t be a positive integer and $H_i = K_{n-mi} - \frac{p}{2q}[(n-im)^2 - s_i]$, where $i = 1, 2, \ldots, t$ and $(n-im)^2 \equiv s_i (mod\ 2q)$. Then for any rational number $0 \le p/q \le 1$, there exists a σ-unreal graph sequence $\overline{H_1 \cup H}$, $\overline{H_2^2(H_v)}$, $\overline{H_3^3(H_v)}$, $\ldots$, $\overline{H_t^t(H_v)}$ such that $\eta(\overline{H_1 \cup H}) \to p/q$ and $\eta(\overline{H_i^i(H_v)}) \to p/q$ as $n \to \infty$, where $i = 2, 3, \ldots, t$.* $\square$

Remarks

In this chapter we studied some properties of the roots of adjoint polynomials.

Firstly, we gave two basic equalities in Theorems 3.2.1 and 3.2.4. Theorem 3.2.1 established an equality between the minimum real roots of the adjoint polynomial of a tree with the largest roots of its characteristic polynomial. Theorem 3.2.4 also established an equality between the minimum real roots of the adjoint polynomial of a graph without triangles with those of the adjoint polynomial of its path-tree. By using these theorems, we translated some results on the largest roots of characteristic polynomials of trees to those on minimum real roots of their adjoint polynomials, see Theorems 3.2.3 and 3.3.1 and (i) and (ii) of Theorem 3.4.1.

Secondly, we obtained a basic inequality on the minimum real roots of the adjoint polynomials of a graph and of its proper subgraphs in Theorem 3.2.2. By employing Theorem 3.2.2 and the results above, we gave some inequalities on minimum real roots of the adjoint polynomials of some graphs used in Chapter 4, see Theorems 3.2.5 and 3.2.6, and determined all connected graphs such that the minimum real roots of their adjoint polynomials belong to the interval $[-4, 0]$, see Theorem 3.3.2, and to the interval $[-(2+\sqrt{5}), -4)$, see Theorem 3.4.1. The results in Sections 3.2 to 3.4 will play an important role in Chapters 4 and 5.

Finally, in Section 3.5 we gave a method to construct graphs such that their σ-polynomials have at least one complex root. A problem posed in 1994 by Brenti, Royle and Wagner in Canadian Journal of Mathematics was solved in Theorems 3.5.3 and 3.5.4. The results in this section are not used in the coming chapters.

Chapter 4

The Chromaticity of Some Dense Graphs

4.1 Introduction

Since the notion of chromatically unique graphs was first introduced by Chao and Whitehead [8] in 1978, many classes of chromatically unique graphs have been found by employing the chromatic polynomials of graphs [43, 44] and the adjoint polynomials of their complements [58]. In fact, the chromatic polynomials of graphs are more useful than the adjoint polynomials of their complements when studying the chromaticity of sparse graphs, whereas the adjoint polynomials of the complements of graphs are more effective than their corresponding chromatic polynomials when studying the chromaticity of dense graphs. On the other hand, many algebraic properties of the adjoint polynomials, such as the recursive relations, divisibility, reducibility over the rational number field and so on, are very useful in the study of chromaticity of the complement of a sparse graph. Indeed, many classes of chromatically unique graphs have been found by applying these properties, see [29] and [58, 62]. In particular, Liu and Li proved that if $G = \cup_i P_{n_i}$, then $K_n - E(G)$ is χ–unique when P_{n_i} is irreducible [52, 60]. In [58], Liu conjectured that $\overline{P_n}$ is χ-unique if $n \neq 4$ and n is even. Du obtained that if G is a 2-regular graph without C_4 as its subgraph or G is $\cup_{i=1}^{k} P_{n_i}$, where n_i is even and

$n_i \not\equiv 4(mod\ 10)$, then $\overline{G}$ is χ-unique [32]. Ye and Li [79] characterized all adjointly equivalent graphs of P_n for $n \geq 1$. Very recently, Dong, Teo, Little and Hendy in [29] investigated the chromaticity of complements of $H_1 = aK_3 \cup bD_4 \cup \bigcup_{1\leq i\leq s} P_{u_i} \cup \bigcup_{1\leq j\leq t} C_{v_j}$, where $a, b \geq 0, u_i \geq 3, u_i \not\equiv 4(mod\ 5), v_j \geq 4$, and of $H_2 = r_0K_1 \cup r_1K_3 \cup \bigcup a_iP_{2i}$, where $r_0, r_1, a_i, i \geq 0$. They obtained a necessary and sufficient condition for $\overline{H_i}$ to be chromatically unique, where $i = 1, 2$.

Our main goal in this chapter is to investigate the chromaticity of some dense graphs by using results obtained in the preceding two chapters. In Section 4.2 we give some basic results on adjoint polynomials. In Section 4.3, we establish a necessary and sufficient condition for chromatic uniqueness of graphs G with $\delta(G) \geq |V(G)| - 3$ and of the graphs $\overline{\cup_i U_{n_i}}$, where $n_i \geq 6$. In Section 4.4 a necessary and sufficient condition for two graphs H and G with $\beta(G) = \beta(H) \geq -4$ to be adjointly equivalent is obtained. Two conjectures proposed by Dong, Teo, Little and Hendy are solved. In Sections 4.5 to 4.7, we obtain some new results on the adjoint uniqueness of graphs.

4.2 Some basic results

In this section, we introduce some basic results used in this chapter. Let G be a graph with q edges. The *character* (or invariant $R_1(G)$) of a graph G is defined as

$$R_1(G) = \begin{cases} 0, & \text{if } q = 0, \\ b_2(G) - \binom{b_1(G)-1}{2} + 1, & \text{if } q > 0, \end{cases}$$

where $b_1(G)$ and $b_2(G)$ are the second and the third coefficients of $h(G)$, respectively.

Lemma 4.2.1. *([50]) Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$R_1(G) = \sum_{i=1}^{k} R_1(G_i).$$

It is not hard to see that $R_1(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_1(G) = R_1(H)$ if $h(G, x) = h(H, x)$ or $h_1(G, x) = h_1(H, x)$.

Lemma 4.2.2. *([31, 62]) Let G be a connected graph and $e \in E(G)$. Then*

$$R_1(G) = R_1(G-e) - d_G(e) + 1.$$

Lemma 4.2.3. *([31]) Let G be a connected graph with p vertices. Then*
(i) $R_1(G) \leq 1$, and the equality holds if and only if $G \in \{P_p, K_3 | p \geq 2\}$.
(ii) $R_1(G) = 0$ if and only if $G \in \{K_1, C_p, D_p, T_{l_1,l_2,l_3} | p \geq 4, l_i \geq 1, i = 1, 2, 3\}$.
(iii) $R_1(G) = -1$ and $q(G) \geq p(G) + 1$ if and only if $G \in \{F_p | p \geq 6\} \cup \{K_4^-\}$.
(iv) $R_1(G) = -2$ and $q(G) \geq p(G) + 2$ if and only if $G \cong K_4$.

Lemma 4.2.4. *([51, 53]) Let G be a graph with p vertices and q edges. Denote by M the set of vertices of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, d_3, \cdots, d_p)$, then*
(i) $b_0(G) = 1, b_1(G) = q$;
(ii) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum\limits_{i=1}^{p} d_i^2 + N_A(G)$;
(iii) $b_3(G) = \frac{1}{6}q(q^2+3q+4) - \frac{q+2}{2}\sum\limits_{i=1}^{p} d_i^2 + \frac{1}{3}\sum\limits_{i=1}^{p} d_i^3 + \sum\limits_{ij\in E(G)} d_i d_j - \sum\limits_{i\in M} M(i) d_i + (q+2)N_A(G) + N(K_4)$, where $b_0(G), b_1(G), b_2(G), b_3(G)$ are the first four coefficients of $h(G,x)$ and $N(K_4)$ is the number of subgraphs isomorphic to K_4 in G.

Lemma 4.2.5. *([32]) If $m_i \geq 3$ and $m_i \neq 4$, then $\overline{\cup_i C_{m_i}}$ is χ–unique.*

Lemma 4.2.6. *([82]) If $n \geq 2$, then $h_1(P_n)$ is irreducible if and only if $n = 3$ or $n + 1$ is prime.*

4.3 The chromaticity of graphs G with $\delta(G) \geq |V(G)| - 3$

By using some properties of the adjoint polynomials of graphs, the authors in [27–29] and in [51–62] gave many chromatically unique graphs. One can see that most of the chromatically unique graphs are graphs of the form $\overline{\cup_i H_i}$ such that $H_i \in \{K_1, P_n, C_m | n \geq 2, m \geq 3\}$ for any i. However, they did not give any sufficient and necessary condition for all graphs of the form $\overline{\cup_i H_i}$ to be χ-unique. In this section, by using the fact that $\beta(\overline{G}) = \beta(\overline{H})$ if $G \sim H$, we obtain a sufficient and necessary condition for all graphs of the form $\overline{\cup_i H_i}$ to be χ-unique. We also obtain a sufficient and necessary condition for all graphs of the form $\overline{\cup_i U_{n_i}}$ to be χ-unique, where $n_i \geq 6$.

From Lemma 3.2.5, it is easy to prove the following Lemma.

Lemma 4.3.1. *([81]) (i)* $(x+3) \nmid h_1(P_{2n})$;
(ii) For $n \geq 1, m \geq 4$, $(h_1(C_m), h_1(P_{2n})) = 1$;
(iii) For $n_1 \geq 3, n_2 \geq 4$, $h_1(P_{n_1})h_1(C_{n_2}) = h_1(P_{n_1+n_2})$ *if and only if* $n_2 = n_1 + 1$;
(iv) All the roots of $h_1(P_n)$ *and* $h_1(C_m)$ *are simple.*

By Theorem 1.2.4, one can check the following results: $h(C_4) = h(D_4)$, $h(P_4) = h(K_1 \cup C_3)$, $h(P_2)h(C_6) = h(P_3)h(D_5)$, $h(P_2)h(C_9) = h(P_5)h(D_6)$ and $h(P_2)h(C_{15}) = h(P_5)h(D_7)h(C_5)$. So, by Corollary 3.2.1 and Lemma 4.3.1, it is easy to prove the following lemma.

Lemma 4.3.2. *(i)* $\beta(C_k) = \beta(P_{2k-1})$ *for* $k \geq 4$ *and* $\beta(C_3) = \beta(P_4)$;
(ii) $\beta(D_4) = \beta(C_4) = \beta(P_7)$;
(iii) $\beta(D_5) = \beta(C_6) = \beta(P_{11})$;
(iv) $\beta(D_6) = \beta(C_9) = \beta(P_{17})$;
(v) $\beta(D_7) = \beta(C_{15}) = \beta(P_{29})$.

By Theorem 1.2.5, $\beta(D_n) = \beta(T_{1,2,n-3})$ and $\beta(C_n) = \beta(T_{1,1,n-2})$, for $n \geq 4$. By Lemma 4.3.2, we have that, if $H \in \{K_1\} \cup \{D_n, T_{1,2,n-3} | n = 4,5,6,7\} \cup \{C_n, T_{1,1,n-2} | n \geq 4\}$, then there exists an integer k, $k \geq 4$, such that $\beta(H) = \beta(P_{2k-1})$. By Theorem 3.2.2, $\beta(P_n) < \beta(P_{n-1})$, for $n \geq 2$. So, we have

Corollary 4.3.1. *Let* $H \in \{K_1\} \cup \{D_n, T_{1,2,n-3} | n = 4,5,6,7\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1$ *and let* $\beta(H) = \beta(P_{2m})$. *If* $m \geq 1$ *and* $m \neq 2$, *then* $H \cong P_{2m}$.

Lemma 4.3.3. *Let* $G = t_1P_2 \cup t_2P_3 \cup t_3P_5 \cup t_4C_3$. *Then* G *is adjointly unique.*

Proof. Let H be a graph such that $h(H) = h(G)$ and $H = \cup_i H_i$. We prove $H \cong G$.

By Corollary 3.3.1, we have

$$H_i \in \{K_1, P_2, P_3, P_4, P_5, C_3, T_{1,1,1}\}.$$

Denote the number of K_1, P_2, P_3, P_4, P_5, C_3 and $T_{1,1,1}$ in H by $m_0, m_1, m_2, m_3, m_4, m_5$ and m_6, respectively. By Lemmas 4.2.1 and 4.2.3, we have

$$R_1(H) = R_1(G) = m_1 + m_2 + m_3 + m_4 + m_5 = t_1 + t_2 + t_3 + t_4.$$

Hence

$$m_1 + m_2 + m_3 + m_4 + m_5 = t_1 + t_2 + t_3 + t_4.$$

Since $h_1(C_3)$ is irreducible over the rational number field and $h_1(P_4) = h_1(C_3)$, we have $m_3 + m_5 = t_4$ and $m_1 + m_2 + m_4 = t_1 + t_2 + t_3$, by Lemma 4.2.3 . As $p(G) - q(G) = t_1 + t_2 + t_3$, $p(H) - q(H) = m_0 + m_1 + m_2 + m_3 + m_4 + m_6$ and $p(G) - q(G) = p(H) - q(H)$, we have $m_0 + m_3 + m_6 = 0$. This implies that $m_0 = m_3 = m_6 = 0$ and $m_5 = t_4$. Therefore,

$$H_i \in \{P_2, P_3, P_5, C_3\}.$$

By Corollary 3.2.1 and Lemma 4.3.2, we have

$$\beta(P_5) < \beta(C_3) < \beta(P_3) < \beta(P_2).$$

Comparing the minimum real roots of $h(G)$ with those of $h(H)$, we get $H \cong G$.

□

Theorem 4.3.1. *Let $n \geq m_i \geq 2$ and $G = K_n - E(\cup_i P_{m_i})$.*
(i) If $n > m$, then G is χ–unique if and only if, for each i, either $m_i \equiv 0 (mod\ 2)$ and $m_i \neq 4$ or $m_i = 3$;
(ii) if $n = m$, then G is χ–unique if and only if, for each i, either $m_i \equiv 0 (mod\ 2)$ and $m_i \neq 4$ or $m_i = 3, 5$, where $m = m_1 + m_2 + \cdots + m_k$, $m_i \geq 2, i = 1, 2, \cdots, k$.

Proof. Note that $\overline{K_n - E(\cup_i P_{m_i})} = lK_1 \cup (\cup_i P_{m_i})$. By Theorem 1.1.1, we have that $K_n - E(\cup_i P_{m_i})$ is χ-unique if and only if $lK_1 \cup (\cup_i P_{m_i})$ is adjointly unique. So, we need only consider the necessary and sufficient condition for $F = lK_1 \cup (\cup_i P_{m_i})$ to be adjointly unique, where $l = n - m$.

We prove the sufficiency of the condition of the theorem. Let H be a graph such that $h(H) = h(F)$ and $H = \cup_i H_i$. Now we prove that $H \cong F$, that is $\cup_i H_i \cong lK_1 \cup (\cup_i P_{m_i})$.

By Theorem 2.1.1, we have

$$\prod_{i=1}^{t} h(H_i) = x^l \prod_{i=1}^{k} h(P_{m_i}). \tag{4.1}$$

Comparing the minimum real roots on both sides of (4.1), by Theorem 3.3.2, we get

$$H_i \in \{K_1, T_{1,2,i}, D_{i+3} | i = 1, 2, 3, 4\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1.$$

Without loss of generality, we assume $m_1 = max\{m_i | i = 1, 2, \cdots, k\}$. From the condition of the theorem, m_i is even if $m_i \geq 6$. By Corollary 3.2.1 and (4.1), we know that $\beta(F) = \beta(P_{m_1})$ and there exists a component in H, say H_1, such that $\beta(H_1) = \beta(H) = \beta(P_{m_1})$. By Corollary 4.3.1, $H_1 \cong P_{m_1}$, for $m_1 \geq 6$. Eliminating a common factor $h(P_{m_1})$ of $h(H)$ and $h(F)$, we have

$$\prod_{i=2}^{t} h(H_i) = x^l \prod_{i=2}^{k} h(P_{m_i}).$$

Repeating elimination procedure for the above equality, we can obtain that, for any $m_i \geq 6$ and m_i is even, there exists a component H_i in H such that $H_i \cong P_{m_i}$. Eliminating all the factors $h(P_{m_i})$ for $m_i \geq 6$ on both sides of equality (4.1), we obtain

$$\prod_{i=t_1}^{t_2} h(H_i) = x^l \prod_{i=k_1}^{k_2} h(P_{m_i}), \tag{4.2}$$

and $m_i \in \{2, 3, 5\}$.

We distinguish two cases:

Case 1. $n = m$.

It is clear that $l = 0$ and $m_i \in \{2, 3, 5\}$. By Lemma 4.3.3, we have $H \cong F$.

Case 2. $n > m$.

In this case, we have $m_i \in \{2, 3\}$. Hence, $H_i \in \{P_2, P_3\}$, by Corollary 3.3.1. By comparing the minimum real roots of the left-hand side with those of the right-hand side in equality (4.2), we have $H \cong F$.

Conversely, note that $h(P_{2n+1}) = h(P_n \cup C_{n+1})$ for $n \geq 3$, $h(P_4) = h(C_3 \cup K_1)$, and $h(P_5 \cup K_1) = h(P_2 \cup T_{1,1,1})$. This shows the necessity of the condition of the theorem. $\square$

Corollary 4.3.2. *Let $n \geq m \geq 2$ and $G = K_n - E(P_m)$.*

(i) If $n > m$, then G is χ–unique if and only if $m \equiv 0(mod\ 2)$ and $m \neq 4$, or $m = 3$;

(ii) if $n = m$, then G is χ–unique if and only if $m \equiv 0(mod\ 2)$ and $m \neq 4$, or $m = 3, 5$.

This corollary gives a positive answer to Du's Problem [32] and Liu's Conjecture [58], which was also done in [29].

Let A, A_i, B, B_i, M, M_i be some multisets with positive integers as their elements for $i = 1, 2$, see §1.2 in [70]. We denote by $A\backslash\{b\}$ the set obtained by deleting an element b from A. For example, let $A = \{a, a, b, b, b, c\}$, then $A\backslash\{b\} = \{a, a, b, b, c\}$.

Lemma 4.3.4. *Let $G = m_1P_2 \cup (\cup_{i\in A_1} P_i) \cup (\cup_{j\in B_1} C_j)$ and $H = m_2P_2 \cup (\cup_{i\in A_2} P_i) \cup (\cup_{j\in B_2} C_j) \cup (\cup_{k\in M_1} D_k)$. If $h_1(G) = h_1(H)$, then $m_1 = m_2 + |M_1|$, where $i \geq 3, j \geq 4, k \geq 5$.*

Proof. Since $h_1(G) = h_1(H)$, we know that $R_1(G) = R_1(H)$ and $q(G) = q(H)$. By Lemmas 4.2.1 and 4.2.3, we have $m_1 + |A_1| = m_2 + |A_2|$ and $p(G) = p(H)$. Let $m_1 + |A_1| = m, p(G) = n$ and $|M_1| = s$. Note that G has n vertices, $n - m$ edges, $2m$ vertices of degree 1 and $N_A(G) = N(K_4) = 0$. By Lemma 4.2.4, we have

$$\begin{aligned} b_3(G) = \ & \tfrac{1}{6}(n-m)\left((n-m)^2 + 3(n-m) + 4\right) - \tfrac{n-m+2}{2}\left(\sum_{i=1}^{n-2m} 2^2 + 2m\right) \\ & + \tfrac{1}{3}\left(\sum_{i=1}^{n-2m} 2^3 + 2m\right) + \sum_{i=1}^{n-3m+m_1} 2^2 + 4(m-m_1) + m_1. \end{aligned}$$

Note that H has $n-m$ edges, n vertices, $2m+s$ vertices of degree 1, s vertices of degree 3, s triangles and $N(K_4)=0$. By Lemma 4.2.4, we have

$$\begin{aligned} b_3(H) = \; & \tfrac{1}{6}(n-m)\left((n-m)^2+3(n-m)+4\right)+\tfrac{1}{3}\left(\sum_{i=1}^{n-2m-2s}2^3+2m+28s\right) \\ & -\tfrac{n-m+2}{2}\left(\sum_{i=1}^{n-2m-2s}2^2+2m+10s\right)+\sum_{i=1}^{n-3m-4s+m_2}2^2 \\ & +4(m-m_2)+m_2+13s+s(n-m+2). \end{aligned}$$

Since $b_3(G)=b_3(H)$, by simplifying we have $m_1=m_2+s=m_2+|M_1|$.

□

Theorem 4.3.2. *Let* $G=(\cup_{i\in A}P_i)\cup(\cup_{j\in B}P_{2j})\cup(\cup_{k\in M}C_k)\cup lC_3$. *Then* $\overline{G}$ *is* χ*–unique if and only if* $1\notin B$ *and* $D=\phi$*, or* $1\in B$ *and* $(M\cap\{6,9,15\})\cup D=\phi$*, where* $D=(\{i|i\in A\}\cap\{k-1|k\in M\})\cup(\{2j|j\in B\}\cap\{k-1|k\in M\})$*,* $i=3$ *or* 5 *if* $i\in A$*,* $k\geq 5$ *if* $k\in M$ *and* $2\notin B$*.*

Proof. By Theorem 1.1.1, it is not difficult to see that we need only prove that the necessary and sufficient condition for G to be adjointly unique is $1\notin B$ and $D=\phi$, or $1\in B$ and $(M\cap\{6,9,15\})\cup D=\phi$.

Let H be a graph such that $h(H)=h(G)$. We prove that $H\cong G$ by induction on $|A|+|B|+|M|+l$.

By Lemma 4.2.5 and Theorem 4.3.1, $H\cong G$ when $|A|+|B|+|M|+l=1$.

Suppose $|A|+|B|+|M|+l=m\geq 2$ and the theorem is true if $|A|+|B|+|M|+l<m$. Let $H=\cup_i H_i$. By Theorem 3.3.2, we have

$$H_i\in\{K_1,T_{1,2,i},D_{i+3}|i=1,2,3,4\}\cup\mathcal{P}\cup\mathcal{C}\cup\mathcal{T}_1. \tag{4.3}$$

Let $n=max\{a|a\in A\cup B'\cup M'\}$, where $B'=\{2j|j\in B\}$ and $M'=\{2k-1|k\in M\}$. We distinguish two cases:

Case 1. $n=2t$ and $t\neq 2$.

By Corollary 3.2.1 and Lemma 4.3.1, there must exist a number $t\in B$ such that $\beta(G)=\beta(P_{2t})$, and there exists a component H_i in H such that $\beta(P_{2t})=\beta(H_i)$. By Corollary 4.3.1, $H_i\cong P_{2t}$. Hence, $H=P_{2t}\cup F$. By the induction hypothesis, we have

$$F\cong[\cup_{\in A}P_i]\cup[\cup_{j\in B\setminus\{t\}}P_{2j}]\cup[\cup_{k\in M}C_k]\cup lC_3.$$

Therefore, $H\cong G$.

Case 2. $n = 2t - 1$.

If $n = 3, 5$, then $M = \phi$, $A = \{3, 5\}$, $l \geq 0$ and $B = \{1\}$. Hence, all components of G are P_2, P_3, P_5 or C_3. By Lemma 4.3.3 , we have $H \cong G$. If $n = 2t-1 \geq 7$, then by Theorem 1.2.5, Corollary 3.2.1, and Lemmas 4.3.1 and 4.3.2 , there exist a number $t \in M$ such that $\beta(G) = \beta(C_t)$ and a component H_i in H such that $\beta(H) = \beta(H_i) = \beta(C_t)$, where $t \geq 4$ and H_i is one of the following graphs

$$P_{2t-1}, C_t, T_{1,1,t-2}, D_4, D_5, D_6, D_7, T_{1,2,i-3},\ 5 \leq i \leq 7.$$

Case 2.1. C_t is a component in H such that $\beta(C_t) = \beta(H)$.

Assume that $H = C_t \cup F$. Then, by the induction hypothesis we have

$$F \cong [\cup_{i \in A} P_i] \cup [\cup_{j \in B} P_{2j}] \cup [\cup_{k \in M \backslash \{t\}} C_k] \cup lC_3.$$

Hence $H \cong G$.

Case 2.2. H contains a component P_{2t-1} such that $\beta(P_{2t-1}) = \beta(H)$.

Without loss of generality, let $H = P_{2t-1} \cup F$. By Lemma 4.3.1, we have

$$h(G, x) = h(H, x) = h(C_t, x)h(P_{t-1}, x)h(F, x).$$

Hence

$$h([\cup_{i \in A} P_i] \cup [\cup_{j \in B} P_{2j}] \cup [\cup_{k \in M \backslash \{t\}} C_k] \cup lC_3) = h(P_{t-1} \cup F).$$

By the induction hypothesis, we have

$$[\cup_{i \in A} P_i] \cup [\cup_{j \in B} P_{2j}] \cup [\cup_{k \in M \backslash \{t\}} C_k] \cup lC_3 \cong P_{t-1} \cup F.$$

Hence $t - 1 \in A \cup B'$ and $t \in M$. This implies $t - 1 \in D$, contradicting $D = \phi$.

Case 2.3. There exists a component $T_{1,1,t-2}$ in H such that $\beta(T_{1,1,t-2)} = \beta(H)$, where $t \geq 4$.

Assume that $H = T_{1,1,t-2} \cup F$. By Theorem 1.2.5, we have

$$h(G, x) = h(H, x) = h(T_{1,1,t-2}, x)h(F, x) = h(C_t, x)[xh(F, x)].$$

So,

$$h([\cup_{i \in A} P_i] \cup [\cup_{j \in B} P_{2j}] \cup [\cup_{k \in M \backslash \{t\}} C_k] \cup lC_3) = h(K_1 \cup H_1).$$

By the induction hypothesis,

$$[\cup_{i\in A}P_i]\cup[\cup_{j\in B}P_{2j}]\cup[\cup_{k\in M\setminus\{t\}}C_k]\cup lC_3\cong K_1\cup F,$$

which is impossible.

Case 2.4. D_i is a component of H and $\beta(D_i)=\beta(H)$ for some i, where $4\le i\le 7$.

If D_4 is a component of H such that $\beta(D_4)=\beta(C_t)$, then $t=4$. This contradicts $4\notin M$. If D_i is a component of H and $\beta(D_i)=\beta(H)=\beta(C_t)$ for some i, where $5\le i\le 7$, then $t=6,9,15$, by Corollary 3.2.1 and Lemma 4.3.2. Hence, according to the condition of the theorem, P_2 is not a component of G. Therefore we have the following claim by Theorems 1.2.5 and 4.3.4.

Claim: H must contain a component $T_{1,1,1}$.

Proof of the Claim. Suppose that H does not contain a component $T_{1,1,1}$. Then, according to (4.3), we can assume that

$$H=m_2P_2\cup(\cup_aP_a)\cup(\cup_bC_b)\cup(\cup_cT_{1,1,c})\cup(\cup_fD_f)\cup(\cup_sT_{1,2,s})\cup rK_1,$$

where $a\ge 3, b\ge 3, c\ge 2, f=4,5,6,7$ and $s=2,3,4$.

Since $h(D_4)=h(C_4)$ and $h(C_3)=h(P_4)$, by Theorem 1.2.5, we have

$$h_1(H)=h_1(m_2P_2\cup(\cup_{i\in A_2}P_i)\cup(\cup_{j\in B_2}C_j)\cup(\cup_{k\in M_1}D_k))$$

and

$$h_1(G)=h_1((\cup_{i\in A}P_i)\cup(\cup_{j\in B\setminus\{1\}}P_{2j})\cup(\cup_{k\in M}C_k)\cup lC_3),$$

where $i\ge 3$ for $i\in A_2$, $j\ge 4$ for $j\in B_2$, and $|M_1|\ge 1$ and $k\ge 5$ for any $k\in M_1$.

Since $h_1(H)=h_1(G)$, by Lemma 4.3.4 we have $m_2+|M_1|=0$, contradicting $|M_1|>0$. This implies that $T_{1,1,1}$ is a component of H if D_i is a component of H, where $i=5,6,7$. This completes the proof of the claim.

Case 2.4.1. D_7 is a component in H and $\beta(D_7)=\beta(H)=\beta(G)$. By Corollary 3.2.1 and Lemma 4.3.2, C_{15} is a component of G and $\beta(C_{15})=\beta(G)$, and the order of a maximum path component (respectively a maximum cycle component) in H is less than 29 (respectively 15). Remembering that

$h(P_2)h(C_{15}) = h(P_5)h(D_7)h(C_5)$, by Lemma 3.2.5 we have

$$h_1(C_{15}) = h_1(D_7)(x+3)(x+2+2cos\frac{\pi}{5})(x+2+2cos\frac{3\pi}{5}),$$

and $h(P_a)$ and $h(C_b)$ do not include the factor $(x+2+2cos\frac{\pi}{5})(x+2+2cos\frac{3\pi}{5})$ when $a \leq 28, b \leq 14$, unless $a = 19$ or $a = 9$, and $b = 5$. Hence, at least one of P_{19}, P_9 and C_5 is a component of H. Since $h(P_{19}) = h(P_4)h(C_5)h(C_{10})$, $h(P_9) = h(P_4)h(C_5)$ and $h(C_{15}) = h(D_7)h(T_{1,1,1})h(C_5)/x$, by the Claim we have

$$h(H) = h(F)h(D_7)h(T_{1,1,1})h(C_5) = h(F \cup K_1)h(C_{15}).$$

Hence,

$$h([\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{15\}}C_k] \cup lC_3) = h(K_1 \cup F).$$

By the induction hypothesis, we have

$$[\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{15\}}C_k] \cup lC_3 \cong K_1 \cup F,$$

which is impossible.

Case 2.4.2. D_6 is a component of H and $\beta(D_6) = \beta(H)$.

By Corollary 3.2.1 and Lemma 4.3.2, $\beta(D_6) = \beta(C_9)$ and C_9 is a component of G. Without loss of generality, we can assume that $H = F \cup D_6 \cup T_{1,1,1}$ by the Claim. As $h(C_9) = h(D_6)h(T_{1,1,1})/x$, we have

$$h(H) = h(F)h(D_6)h(T_{1,1,1}) = h(F \cup K_1)h(C_9).$$

Hence, we obtain

$$h([\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{9\}}C_k] \cup lC_3) = h(K_1 \cup F).$$

By the induction hypothesis, we have

$$[\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{9\}}C_k] \cup lC_3 \cong K_1 \cup F,$$

which is impossible.

Case 2.4.3. D_5 is a component of H and $\beta(D_5) = \beta(H)$.

By Corollary 3.2.1 and Lemma 4.3.2, C_6 is a component of G and $\beta(C_6) = \beta(G)$, and the order of a maximum path component (respectively a maximum cycle component) in H is less than 11(respectively 6). Noticing that $h(P_2)h(C_6) = h(P_3)h(D_5)$, by Lemma 3.2.5, we have

$$h_1(C_6) = (x+2)(x+2+2cos\frac{\pi}{6})(x+2+2cos\frac{5\pi}{6}),$$

and $h_1(P_a)$ and $h_1(C_b)$ do not include the factor $(x+2)$ when $a < 11, b < 6$, unless $a = 3$ or $a = 7$; and only $h_1(P_5)$ includes the factor $(x+3)$. Hence, at least one of P_3 or P_7 is a component of H and P_5 must be a component of G. Since $h(P_7) = h(P_3)h(C_4)$, by the Claim we have

$$h(H) = h(F)h(P_3)h(T_{1,1,1})h(D_5) = h(F)h(P_2)h(C_6)h(T_{1,1,1}),$$

and

$$h(G) = h(G_1)h(P_5)h(C_6).$$

Hence, we have

$$h([\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{6\}}C_k] \cup lC_3) = h(P_2 \cup T_{1,1,1} \cup F).$$

By the induction hypothesis, we get

$$[\cup_{i\in A}P_i] \cup [\cup_{j\in B}P_{2j}] \cup [\cup_{k\in M\setminus\{6\}}C_k] \cup lC_3 \cong P_2 \cup T_{1,1,1} \cup F,$$

which is impossible.

Case 2.5. $T_{1,2,i}$, where $2 \le i \le 4$, is a component of H.

Let $H = T_{1,2,i} \cup F$. We have

$$h(G,x) = h(H,x) = h(T_{1,2,i},x)h(F,x) = h(D_{i+3},x)[xh(F,x)],$$

which is impossible from Case 2.4.

Conversely, if $j = i+1$ and $i \ge 3$, then $h(P_i)h(C_{i+1}) = h(P_{2i+1})$ by Lemma 1.2.5. Note that $h(P_2)h(C_6) = h(P_3)h(D_5), h(P_2)h(C_9) = h(P_5)h(D_6)$ and $h(P_2)h(C_{15}) = h(P_5)h(D_7)h(C_5)$. This shows the necessity of the condition of the theorem. $\square$

It is easy to see that if G is a graph with $\delta(G) \geq p(G) - 3$, then each component of $\overline{G}$ is one of the graphs K_1, P_i and C_j, where $i \geq 2$ and $j \geq 3$. By Theorem 1.2.5, we have that $K_1 \cup K_3 \sim_h P_4$ and $K_1 \cup C_n \sim_h T_{1,1,n-2}$, for $n \geq 4$. Hence the following theorem follows from Theorems 4.3.1 and 4.3.2.

Theorem 4.3.3. *Let $A = \{n|n \equiv 0(mod\ 2)$ and $n \geq 6\}$ and $B = \{n|n \geq 5\}$. For a graph G with p vertices and $\delta(G) \geq p - 3$, we have that G is χ-unique if and only if $\overline{G}$ is one of the following graphs:*
(i) $rK_1 \cup (\bigcup\limits_{1\leq i\leq s} P_{n_i})$ for $r = 0$ and $n_i \in A \cup \{2, 3, 5\}$, or $r \geq 1$ and $n_i \in A \cup \{2, 3\}$, where $r, s \geq 0$;
(ii) $t_1P_2 \cup (\bigcup\limits_{1\leq i\leq s} P_{n_i}) \cup (\bigcup\limits_{1\leq j\leq t} C_{m_j}) \cup lC_3$ for $t_1 = 0$ and $M = \phi$, or $t_1 \geq 1$ and $(\{6, 9, 15\} \cap B) \cup M = \phi$, where $t_1, l, s, t \geq 0$, $t + l \geq 1$, $n_i \in A \cup \{3, 5\}$, $m_j \in B$ and $M = (A \cup \{3, 5\}) \cap \{n - 1|n \in B\}$.

Lemma 4.3.5. *For any $m \geq 6$ and $n \geq 5$, we have $h(U_m) = x^3(x+4)h(P_{m-4})$ and $h(U_{2n+1}) = h(U_{n+2})h(C_{n-1})$.*

Proof. By Theorem 2.1.2, for $m \geq 6$, we have

$$\begin{aligned} h(U_m) &= xh(T_{1,1,m-4}) + x^2h(T_{1,1,m-6}) \\ &= x^2h(P_{m-2}) + 2x^3h(P_{m-4}) + x^4h(P_{m-6}) \\ &= x^3(x + 4)h(P_{m-4}). \end{aligned}$$

By Lemma 1.2.5(i), if $n \geq 5$ and $m = 2n + 1$, then

$$h(U_{2n+1}) = x^3(x + 4)h(P_{n-2})h(C_{n-1}) = h(U_{n+2})h(C_{n-1}). \qquad \square$$

From Theorem 4.3.1(i), we have

Lemma 4.3.6. *Let $n_i = 3$ or $n_i \geq 6$ and n_i is even. If*

$$\prod_{i=1}^{m_1} h_1(P_{n_i}) = \prod_{j=1}^{m_2} h_1(H_j),$$

then $m_1 = m_2$ and $\cup_{i=1}^{m_1} P_{n_i} \cong \cup_{j=1}^{m_2} H_j$, where H_j is connected and $H_j \ncong K_1$ for each j.

Theorem 4.3.4. *Let $n_i \geq 6$. Then $\overline{\cup_{i=1}^{m} U_{n_i}}$ is χ-unique if and only if $n_i = 7$ or $n_i \geq 10$ and n_i is even, where $i = 1, 2, \cdots, m$.*

Proof. Let $G = \cup_{i=1}^{m} U_{n_i}$. Suppose that $h(H) = h(G)$ and let $H = \cup_{j=1}^{m_1} H_j$. By Theorem 1.1.1, we only need prove $H \cong G$. By Theorem 2.1.1, we have

$$\prod_{i=1}^{m} h(U_{n_i}) = \prod_{j=1}^{m_1} h(H_j). \tag{4.4}$$

By Theorem 3.3.2, we have

$$H_j \in \{T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2, P_2), K_4^-, D_8, K_1\} \cup$$
$$\{T_{1,2,i} | i = 2, 3, 4, 5\} \cup \{D_i | i = 4, 5, 6, 7\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{U}.$$

By calculation, we obtain the following:
$h_1(C_3(P_2, P_2)) = h_1(C_4(P_2)) = h_1(K_4^-) = h_1(P_2)h_1(K_{1,4})$,
$h_1(D_8) = h_1(T_{1,2,5}) = h_1(P_2)h_1(P_4)h_1(K_{1,4})$,
$h_1(T_{1,3,3}) = h_1(P_2)h_1(P_3)h_1(K_{1,4})$,
$h_1(T_{2,2,2}) = h_1^2(P_2)h_1(K_{1,4})$.

Since $h_1(K_{1,4}) = x + 4$, eliminating all the factors $x + 4$ and x on both sides of (4.4), we obtain, from Lemma 4.3.5, that

$$\prod_{i=1}^{m} h_1(P_{n_i - 4}) = \prod_{j=1}^{m_2} h_1(H'_j), \quad m_2 \le m_1,$$

and

$$H'_j \in \{T_{1,2,i} | i = 2, 3, 4\} \cup \{D_i | i = 4, 5, 6, 7\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1.$$

Note that $n_i - 4 = 3$ or $n_i - 4 \ge 6$ and $n_i - 4$ is even. By Lemma 4.3.6, we have

$$\cup_{i=1}^{m} P_{n_i - 4} \cong \cup_{j=1}^{m_2} H'_j. \tag{4.5}$$

Hence $H_j \in \{K_{1,4}\} \cup \mathcal{P} \cup \mathcal{U}$ and H must have exactly m components $H_1, H_2, \cdots, H_m$ such that $\beta(H_i) = -4$ and $m \le m_1$. For each component H_j, we have $q(H_j) - p(H_j) = -1, j = 1, 2, \cdots, m_1$. Hence $q(H) - p(H) = -m_1$. Since $q(G) - p(G) = -m$ and $q(H) - p(H) = q(G) - p(G)$, we have $m = m_1 = m_2$ and $H_j \in \mathcal{U}, j = 1, 2, \cdots, m$. By (4.4) and (4.5), we have $G \cong H$.

Note that $h(U_6) = h(K_4^-)h(2K_1)$, $h(U_9) = h(K_1)h(K_{1,3})h(K_4^-)$ and $h(U_8) = h(C_3)h(K_{1,4})$. So, the necessity of the condition of the theorem follows from Lemma 4.3.5 immediately. □

Corollary 4.3.3. *Let $n \ge 6$. Then $\overline{U_n}$ is χ-unique if and only if $n = 7$ or $n \ge 10$ and n is even.*

4.4 A solution to two conjectures on adjointly equivalent graphs

In this section, we obtain a necessary and sufficient condition for two graphs H and G with $\beta(G) = \beta(H) \geq -4$ to be adjointly equivalent. Moreover we give a negative answer to Conjectures 4.4.1 and 4.4.2.

It is an interesting problem to determine $[G]$ for a given graph G. From Theorem 1.1.1, it is not difficult to see that the goal of determining $[G]$ for a given graph G can be realized by determining $[\overline{G}]_h$. In [29], Dong, Teo, Little and Hendy determined all adjointly equivalent classes of graphs $r_0K_1 \cup r_1K_3 \cup \bigcup_{1\leq i\leq s} P_{2l_i}$ for $r_0, r_1 \geq 0, l_i \geq 1$ and obtained a necessary and sufficient condition for two graphs H and G in $\mathcal{G}_1$ to be adjointly equivalent, where

$$\mathcal{G}_1 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\leq i\leq s} P_{u_i} \cup \bigcup_{1\leq j\leq t} C_{v_j} | a, b, r, t \geq 0, u_i \geq 3, u_i \not\equiv 4 (mod\ 5), v_j \geq 4 \right\}.$$

Let

$$\mathcal{G}_2 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1\leq i\leq s} P_{u_i} \cup \bigcup_{1\leq j\leq t} C_{v_j} | a, b \geq 0, u_i \geq 3, v_j \geq 4 \right\}$$

and

$$\mathcal{G}_3 = \left\{ rK_1 \cup \bigcup_{1\leq j\leq t} C_{v_j} | r, t \geq 0, v_, \geq 4 \right\}.$$

In fact, it is not easy to determine the equivalence class of each graph in $\mathcal{G}_i$ for $i = 1, 2, 3$. So, the authors in [29] proposed the following problem: For a set $\mathcal{G}$ of graphs, determine

$$min_h \mathcal{G} = \bigcup_{G\in\mathcal{G}} [G]_h,$$

where $min_h \mathcal{G}$ is called *the adjoint closure* of $\mathcal{G}$.

Dong, Teo, Little and Hendy proposed the following problem and conjectures.

Problem 4.4.1. *([29]) Determine $min_h(\mathcal{G}_2)$.*

Conjecture 4.4.1. *([29]) The following set equality holds.*

$$min_h(\mathcal{G}_2) \equiv \left\{ rK_1 \cup aK_3 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le i\le s} P_{u_i} \right.$$

$$\left. \cup \bigcup_{1\le j\le t} C_{v_j} | r,a,b,t,m,s \ge 0, r_i \ge 2, u_i \ge 3, v_j \ge 4, m+r \le a \right\}.$$

Conjecture 4.4.2. *([29]) The following set equality holds.*

$$min_h(\mathcal{G}_3) \equiv \left\{ rK_1 \cup bD_4 \cup \bigcup_{1\le i\le m} T_{1,1,r_i} \cup \bigcup_{1\le j\le t} C_{v_j} | r,b,m,t \ge 0, r_i \ge 2, v_j \ge 4 \right\}.$$

Let

$$\mathcal{F}_1 = \{K_1, T_{1,2,i}, D_{i+3} | i = 1,2,3,4\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_1$$

and

$$\mathcal{F}_2 = \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), C_3(P_2,P_2), K_4^-, D_8\} \cup \mathcal{U}.$$

By Theorem 3.3.2, it is obvious that $\mathcal{F}_1$ is the set of all connected graphs with $\beta(G) > -4$ and $\mathcal{F}_2$ is the set of all connected graphs with $\beta(G) = -4$. Take $\mathcal{Y}_1 = \{\bigcup G_i | G_i \in \mathcal{F}_1\}$ and $\mathcal{Y}_2 = \{\bigcup G_i | G_i \in \mathcal{F}_1 \cup \mathcal{F}_2\}$. Clearly, both $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are adjointly closed.

From Theorems 2.1.2, 1.2.4 and 1.2.5, one can check that each pair of the graphs in $\mathcal{R}_1$ is adjointly equivalent. We call a pair of adjointly equivalent graphs *an adjointly equivalent transform*. For example, $P_{2n+1} \sim_h P_n \cup C_{n+1}$ is an adjointly equivalent transform.

$\mathcal{R}_1 = \{P_{2n+1} \sim_h P_n \cup C_{n+1}, K_1 \cup C_3 \sim_h P_4, T_{1,1,m-2} \sim_h K_1 \cup C_m, T_{1,2,s-3} \sim_h K_1 \cup D_s, K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}, D_6 \cup T_{1,1,1} \sim_h K_1 \cup C_9, D_7 \cup T_{1,1,1} \cup C_5 \sim_h K_1 \cup C_{15}, P_3 \cup D_5 \sim_h P_2 \cup C_6 | n \ge 3, m \ge 4, s \ge 4\}$.

Theorem 4.4.1. *(i) There exists a graph G in $min_h\mathcal{G}_2$ such that G contains each graph in $\mathcal{F}_1$ as its component and $min_h\mathcal{G}_2 \subseteq \mathcal{Y}_1$.*

(ii) Let $\mathcal{F}_3 = \{K_1\} \cup \{T_{1,2,i}, D_{i+3} | i = 1,3,4\} \cup \{C_i | i \ge 4\} \cup \mathcal{T}_1$ and $\mathcal{Y}_3 = \{\bigcup G_i | G_i \in \mathcal{F}_3\}$. Then there exists a graph G in $min_h\mathcal{G}_3$ such that G contains each graph in $\mathcal{F}_3$ as its component and $min_h\mathcal{G}_3 \subseteq \mathcal{Y}_3$.

Proof. (i) Observing the minimum real roots of each graph in $min_h\mathcal{G}_2$, by Theorem 3.3.2 we have $min_h\mathcal{G}_2 \subseteq \mathcal{Y}_1$.

On the other hand, since $P_4 \sim_h K_3 \cup K_1$, $P_9 \sim_h P_4 \cup C_5$ and $P_{2n+1} \sim_h P_n \cup C_{n+1}$, for $n \geq 3$, we have that K_1 and K_3 are components of some graphs in $min_h\mathcal{G}_2$. Note that $K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}$, $P_2 \cup C_6 \sim_h P_3 \cup D_5$, $K_1 \cup C_9 \sim_h D_6 \cup T_{1,1,1}$, $K_1 \cup C_{15} \sim_h D_7 \cup T_{1,1,1} \cup C_5$, $K_1 \cup C_n \sim_h T_{1,1,n-2}$, for $n \geq 4$, and $K_1 \cup D_m \sim_h T_{1,2,m-3}$, for $m \geq 4$. Hence (i) holds.

(ii) Let $G \in \mathcal{G}_3$ and $H \sim_h G$. From Theorem 3.3.2, we have $H \in \mathcal{Y}_1$. From Theorem 2.1.1 and Lemmas 4.2.1 and 4.2.3, we have $R_1(G) = R_1(H) = 0$, so none of components of H is isomorphic to C_3, or P_n, for $n \geq 2$. Hence we know that each component of H is one of the following graphs:

$$K_1, D_i, T_{1,1,r}, C_j, T_{1,2,s},$$

where $i = 4, 5, 6, 7$; $r \geq 1$; $j \geq 4$ and $s = 2, 3, 4$.

Suppose that D_5 or $h(T_{1,2,2})$ is a component of H. Since $(x+1)|h(D_5) = h(T_{1,2,2})/x$, we have that $(x+1)|h(H)$ and $x+1|h(G)$, which contradicts the fact that $h(P_2) \not| h(C_{v_j})$ for all $v_j \geq 4$, by Lemma 4.3.1(ii). So, $H \in \mathcal{Y}_3$. Thus $min_h\mathcal{G}_3 \subseteq \mathcal{Y}_3$.

Since $C_4 \sim_h D_4$, $K_1 \cup C_9 \sim_h D_6 \cup T_{1,1,1}$, $K_1 \cup C_{15} \sim_h D_7 \cup T_{1,1,1} \cup C_5$, $K_1 \cup C_n \sim_h T_{1,1,n-2}$ and $K_1 \cup D_n \sim_h T_{1,2,n-3}$, we have that there exists a graph G in $min_h\mathcal{G}_3$ such that G contains the following all components:

$$K_1, D_i, T_{1,1,r}, C_j, T_{1,2,s},$$

where $i = 4, 6, 7$; $r \geq 1$; $j \geq 4$ and $s = 3, 4$. □

It is not difficult to see that Theorem 4.4.1 gives negative answers to Conjectures 4.4.1 and 4.4.2.

Let $\mathcal{G}_4 = \{rK_1 \cup \bigcup_{1 \leq i \leq s} P_{u_i} | u_i \geq 2, r \geq 1\}$. With a proof similar to that of Theorem 4.4.1, we prove easily the following result.

Theorem 4.4.2. *There exists a graph G in $min_h\mathcal{G}_4$ such that G contains each graph in $\mathcal{F}_1$ as its component and $min_h\mathcal{G}_4 \subseteq \mathcal{Y}_1$.*

Lemma 4.4.1. *Let* $G_i, H_j \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2a} | n \geq 4, a \geq 1\}$. *If* $\bigcup\limits_{1 \leq i \leq m} G_i \sim_h \bigcup\limits_{1 \leq j \leq t} H_j$, *then* $\bigcup\limits_{1 \leq i \leq m} G_i \cong \bigcup\limits_{1 \leq j \leq t} H_j$.

Proof. By Theorem 2.1.1,

$$\prod_{i=1}^{m} h(G_i) = \prod_{j=1}^{t} h(H_j) \tag{4.6}$$

and $G_i, H_j \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2a} | n \geq 4, a \geq 1\}$.

By induction on m we show that $\bigcup\limits_{1 \leq i \leq m} G_i \cong \bigcup\limits_{1 \leq j \leq t} H_j$.

When $m = 1$, it is clear that

$$h(G_1) = \prod_{j=1}^{t} h(H_j)$$

and $G_1, H_j \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2a} | n \geq 4, a \geq 1\}$. Thus there exists a component in $\bigcup\limits_{1 \leq j \leq t} H_j$, say H_1, such that $\beta(G_1) = \beta(H_1)$. From Lemmas 4.3.1 and 4.3.2 and Corollary 3.2.1, by comparing $\beta(G_1)$ with $\beta(H_1)$, we know that $G_1 \cong H_1$. Moreover, $m = t = 1$ and the theorem holds for $m = 1$.

Suppose that $\bigcup\limits_{1 \leq i \leq m} G_i \cong \bigcup\limits_{1 \leq j \leq t} H_j$ for $m = k - 1$ and $k \geq 2$. When $m = k$, from (4.6) it follows that

$$\prod_{i=1}^{k} h(G_i) = \prod_{j=1}^{t} h(H_j) \tag{4.7}$$

and

$$G_i, H_j \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2a} | n \geq 4, a \geq 1\}$$

for $1 \leq i \leq k$ and $1 \leq j \leq t$.

Now, we consider the minimum real roots on both sides of (4.7). Denote by $\beta(right)$ and $\beta(left)$, respectively, the minimum real roots of the right-hand side and of the left-hand side of (4.7). Without loss of generality, we assume that $\beta(left) = \beta(G_k)$. We distinguish the following cases:

Case 1. $G_k \cong C_n$ for some $n \geq 4$.

Clearly, H has a component, say H_t, such that $\beta(C_n) = \beta(H_t)$. So, by Lemmas 4.3.1 and 4.3.2, $H_t \cong C_n$. From (4.7) we get

$$\prod_{i=1}^{k-1} h(G_i) = \prod_{j=1}^{t-1} h(H_j) \tag{4.8}$$

and $G_i, H_j \in \{K_1, T_{1,1,1}, P_3, C_n, P_{2a} | n \geq 4, a \geq 1\}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq t-1$. By (4.8) and the induction hypothesis, $\bigcup\limits_{1\leq i\leq k-1} G_i \cong \bigcup\limits_{1\leq j\leq t-1} H_j$, and so, $G \cong H$.

Case 2. $G_k \in \{P_4, P_3, T_{1,1,1}, P_2, K_1\}$.

Since $\beta(P_6) < \beta(T_{1,1,1}) < \beta(P_4) < \beta(P_3) < \beta(P_2) < \beta(K_1)$ and $\beta(C_4) < \beta(T_{1,1,1})$, by Lemma 4.3.2 one can see that $G_i, H_j \in \{K_1, P_2, P_3, P_4, T_{1,1,1}\}$ for all $1 \leq i \leq k$ and $1 \leq j \leq t$. Clearly, the theorem holds.

Case 3. $G_k \cong P_{2\alpha}$ for some α.

Obviously, $\alpha \geq 3$. Then, it is not difficult to see that H has a component, say H_t, such that $\beta(P_{2\alpha}) = \beta(H_t)$. So, by Lemmas 4.3.1 and 4.3.2, we have $H_t \cong P_{2\alpha}$. By the induction hypothesis, $\bigcup\limits_{1\leq i\leq k-1} G_i \cong \bigcup\limits_{1\leq j\leq t-1} H_j$. So, $G \cong H$.

□

Suppose that G and H are two graphs. We construct a pair of graphs G^* and H^*, respectively from G and H, by the following steps:

O_1: We construct a pair of graphs G' and H', respectively from G and H, by replacing each component Y by an adjointly equivalent transform in R until none of components is isomorphic to Y, where $Y \in \{P_{2n+1}, D_4, T_{1,1,m}, T_{1,2,s} |n \geq 3, m \geq 2, s \geq 2\}$ and $R \in \{P_{2n+1} \sim_h P_n \cup C_{n+1}, D_4 \sim_h C_4, T_{1,1,m} \sim_h K_1 \cup C_{m+2}, T_{1,2,s} \sim_h K_1 \cup D_{s+3} | n \geq 3, m \geq 2, s \geq 2\}$;

O_2: We denote by a_1, a_2, a_3, a_4 and a_5, respectively, the number of components C_3, P_5, D_5, D_6 and D_7 of G'. We denote by b_1, b_2, b_3, b_4 and b_5, respectively, the number of components C_3, P_5, D_5, D_6 and D_7 of H'. Let $x_1 = max\{a_1 + a_2, b_1 + b_2\}, x_2 = max\{a_3, b_3\}$, $x_3 = max\{a_4 + a_5, b_4 + b_5\}$ and $x_4 = max\{a_5, b_5\}$. Then we take $G'' = G' \cup x_1K_1 \cup x_2P_3 \cup x_3T_{1,1,1} \cup x_4C_5$ and $H'' = H' \cup x_1K_1 \cup x_2P_3 \cup x_3T_{1,1,1} \cup x_4C_5$.

O_3: We construct a pair of graphs G^* and H^*, respectively from G'' and H'', by replacing each component Y' by an adjointly equivalent transform in R' until none of components is isomorphic to Y', where $Y' \in \{K_1 \cup C_3, K_1 \cup P_5, D_6 \cup T_{1,1,1}, D_7 \cup T_{1,1,1} \cup C_5, P_3 \cup D_5\}$ and $R' \in \{K_1 \cup C_3 \sim_h P_4, K_1 \cup P_5 \sim_h P_2 \cup T_{1,1,1}, D_6 \cup T_{1,1,1} \sim_h K_1 \cup C_9, D_7 \cup T_{1,1,1} \cup C_5 \sim_h K_1 \cup C_{15}, P_3 \cup D_5 \sim_h P_2 \cup C_6\}$.

Here we point out that the above operations are valid only for pairs of graphs, but not for a single graph. For convenience, the pair of graphs G^* and H^* are said to be obtained from G and H by the operation OP_1, denoted simply by $< G, H > \xrightarrow{OP_1} < G^*, H^* >$.

Theorem 4.4.3. *Let* $G, H \in \mathcal{Y}_1$ *and* $< G, H > \xrightarrow{OP_1} < G^*, H^* >$. *Then* $G \sim_h H$ *if and only if* $G^* \cong H^*$.

Proof. Suppose that $G, H \in \mathcal{Y}_1$ and $G \sim_h H$. It is clear that $G' \sim_h G \sim_h H \sim_h H'$ and $G^* \sim_h G'' \sim_h H'' \sim_h H^*$. So, by steps O_2 and O_3, one can see that each component of G^* and H^* is one of the following graphs:

$$K_1, T_{1,1,1}, P_3, C_n, P_{2i}, \quad n \geq 4, i \geq 1.$$

By Lemma 4.4.1, $G^* \cong H^*$.

Conversely, suppose that $G^* \cong H^*$. Then $G'' \sim_h H''$ and $G' \sim_h H'$. Thus $G \sim_h H$. □

From Theorems 2.1.2 and 1.2.4 and Lemmas 3.2.7 and 4.3.5, it is not hard to obtain the adjointly equivalent transforms in $\mathcal{R}_2$.

$\mathcal{R}_2 = \{K_1 \cup U_n \sim_h P_{n-4} \cup K_{1,4}, 2K_1 \cup T_{1,2,5} \sim_h P_2 \cup P_4 \cup K_{1,4}, 2K_1 \cup T_{2,2,2} \sim_h 2P_2 \cup K_{1,4}, 2K_1 \cup T_{1,3,3} \sim_h P_2 \cup P_3 \cup K_{1,4}, 2K_1 \cup C_3(P_2, P_2) \sim_h P_2 \cup K_{1,4}, 2K_1 \cup C_4(P_2) \sim_h P_2 \cup K_{1,4}, 3K_1 \cup K_4^- \sim_h P_2 \cup K_{1,4}, 3K_1 \cup D_8 \sim_h P_2 \cup P_4 \cup K_{1,4}\}$.

Suppose $G, H \in \mathcal{Y}_2$. Similar to OP_1, $\widehat{G}$ and $\widehat{H}$ are said to be obtained from G and H by the operation OP_2, denoted by $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >$, if the pair of graphs $\widehat{G}$ and $\widehat{H}$ can be obtained, respectively from G and H, by the following steps:

O_4: Let $y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8$ be respectively the number of components K_1, $U_n, T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, C_3(P_2, P_2), C_4(P_2), K_4^-, D_8$ of G, and let

$y'_0, y'_1, y'_2, y'_3, y'_4$, y'_5, y'_6, y'_7, y'_8 denote respectively the number of components K_1, U_n, $T_{1,2,5}$, $T_{2,2,2}$, $T_{1,3,3}$, $C_3(P_2, P_2)$, $C_4(P_2)$, K_4^-, D_8 of H. Suppose that $y = max\{y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 3y_7 + 3y_8 - y_0, y'_1 + 2y'_2 + 2y'_3 + 2y'_4 + 2y'_5 + 2y'_6 + 3y'_7 + 3y'_8 - y'_0\}$. Take $G_0 = G \cup yK_1$ and $H_0 = H \cup yK_1$;

O_5: We construct a pair of graphs G_1 and H_1, respectively from G_0 and H_0, by replacing each component Y'' by an adjointly equivalent transform in R_2 until none of the components is isomorphic to Y'', where $Y'' \in \{K_1 \cup U_n, 2K_1 \cup T_{1,2,5}, 2K_1 \cup T_{2,2,2}, 2K_1 \cup T_{1,3,3}, 2K_1 \cup C_3(P_2, P_2), 2K_1 \cup C_4(P_2), 3K_1 \cup K_4^-, 3K_1 \cup D_8 | n \geq 6\}$. In fact, G_1 and H_1 contain none of the following components: U_n, $T_{1,2,5}$, $T_{2,2,2}$, $T_{1,3,3}$, $C_3(P_2, P_2)$, $C_4(P_2)$, K_4^- and D_8.

O_6: Let s_1 and s_2 be respectively the number of components $K_{1,4}$ of G_1 and H_1. Take $s = min\{s_1, s_2\}$. By deleting $sK_{1,4}$ from G_1 and H_1, we obtain graphs G_2 and H_2. Note that if $G \sim_h H$, then $s_1 = s_2$ and $G_2, H_2 \in \mathcal{Y}_1$.

O_7: By using OP_1, we obtain the pair of graphs $\widehat{G}$ and $\widehat{H}$, respectively from G_2 and H_2; that is, $< G_2, H_2 > \xrightarrow{OP_1} < \widehat{G}, \widehat{H} >$.

With a proof similar to that of Theorem 4.4.3, we get the following result.

Theorem 4.4.4. *Let $G, H \in \mathcal{Y}_2$ and $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >$. Then $G \sim_h H$ if and only if $\widehat{G} \cong \widehat{H}$.*

By Theorems 4.4.3 and 4.4.4, we have

Theorem 4.4.5. *(i) For any graph $G \in \mathcal{Y}_1$, $[G]_h = \{H \in \mathcal{Y}_1 | H^* \cong G^*$ and $< G, H > \xrightarrow{OP_1} < G^*, H^* >\}$;*
(ii) For any graph $G \in \mathcal{Y}_2$, $[G]_h = \{H \in \mathcal{Y}_2 | \widehat{H} \cong \widehat{G}$ and $< G, H > \xrightarrow{OP_2} < \widehat{G}, \widehat{H} >\}$.

4.5 The adjoint uniqueness of the union of T-shaped trees

We call the tree $T_{a,b,c}$ T-shaped trees. In this section, we study the chromatic uniqueness of the complements of $(\cup_i C_{n_i}) \cup (\cup_i D_{m_j}) \cup (\cup_{a,b} T_{1,a,b})$ and of $rK_1 \cup \bigcup_{a,b} T_{1,a,b}$.

Lemma 4.5.1. *Let $f_i(x)$ be a polynomial in x with integral coefficients. If $h_1(P_m) \not| f_i(x)$, for $m \geq 2$ and for $i = 1, 2, \cdots, k$, then there is no n, $n \geq 2$, such that $h_1(P_n) | \prod_{i=1}^{k} f_i(x)$.*

Proof. Suppose that there is a n, $n \geq 2$, such that $h_1(P_n) | \prod_{i=1}^{k} f_i(x)$. Clearly, $n + 1 \geq 3$. So, there is a n_1 such that $n + 1 = (n_1 + 1)n_2$ and $n_1 + 1 = 4$ or $n_1 + 1$ is prime. From Lemma 4.2.6, $h_1(P_3)$ and $h_1(P_{n_1})$ are irreducible and $h_1(P_3)|h_1(P_n)$ or $h_1(P_{n_1})|h_1(P_n)$. Thus, $h_1(P_3)| \prod_{i=1}^{k} f_i(x)$ or $h_1(P_{n_1})| \prod_{i=1}^{k} f_i(x)$, which implies that there is a i such that $h_1(P_3)|f_i(x)$ or $h_1(P_{n_1})|f_i(x)$. This contradicts the condition of the lemma. □

Theorem 4.5.1. *Let $n_i \geq 5$ and $m_j \geq 9$ for every integers i and j, and let $3 \leq l_1 \leq 10$ and $l_1 \leq l_2$. Let $G = (\cup_i C_{n_i}) \cup (\cup_j D_{m_j}) \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$. If $h(P_n) \not| h(C_{n_i})$, $h(P_n) \not| h(D_{m_j})$ and $h(P_n) \not| h(T_{1,l_1,l_2})$, for all $n \geq 2$, then $\overline{G}$ is χ–unique if and only if $l_2 \neq 2l_1 + 5$ and $(l_1, l_2) \neq (n_i - 1, n_i)$, for all i.*

Proof. From Theorem 1.1.1, we only need consider the necessary and sufficient conditions for G to be adjointly unique.

Let H be a graph such that $h(H) = h(G)$. Suppose that $H = \cup_i H_i$ and each H_i is connected. It is sufficient to prove that $H \cong G$.

By Theorem 2.1.1 and Lemmas 4.2.1 and 4.2.3,

$$\prod_i h(H_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \tag{4.9}$$

and

$$\sum_i R_1(H_i) = \sum_i R_1(C_{n_i}) + \sum_j R_1(D_{m_j}) + \sum_{l_1,l_2} R_1(T_{1,l_1,l_2}) = 0. \tag{4.10}$$

By the conditions of the theorem and Lemma 4.5.1, $h(P_n) \not| h(G) = h(H)$, for all $n \geq 2$. Thus, $h(P_n) \not| h(H_i)$, for each i and for $n \geq 2$. From (4.10), Theorem 2.1.1 and Lemmas 4.2.1 and 4.2.3 and $h_1(P_4) = h_1(K_3)$, we have $R_1(H_i) = 0$ for each component H_i in H. Recalling that $h(P_n) \not| h(H_i)$ for each H_i and for $n \geq 2$, by Lemmas 4.2.3 and 3.2.8, we have

$$H_i \in \{C_n, D_m, T_{a,b,c}, K_1 | n \geq 4, m \geq 4, 1 \leq a \leq b \leq c\} \tag{4.11}$$

and

$$H_i \notin \{T_{1,a,a}, T_{1,b,b+3}, T_{2,2,c}, T_{2,3,3} | a \geq 2, b \geq 1, c \geq 2\}. \tag{4.12}$$

By Theorem 1.2.5, $\beta(C_n) = \beta(T_{1,1,n-2})$ and $\beta(D_n) = \beta(T_{1,2,n-3})$ for $n \geq 4$. Therefore, by Theorem 3.4.1, $\beta(G) > -(2+\sqrt{5})$. So, $\beta(H) = \beta(G) > -(2+\sqrt{5})$. From Theorem 3.4.1 and (4.11) and (4.12), we have

$$H_i \in \{C_n, D_m, T_{1,b,c}, K_1 | n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b+3\}. \tag{4.13}$$

Note that $h(T_{1,a,2a+5}) = h(T_{1,a+1,a+2})h(C_{a+2})$, for $a \geq 2$. We construct a graph H' from H by replacing each component $T_{1,a,2a+5}$ by two components C_{a+2} and $T_{1,a+1,a+2}$ until none of components is isomorphic to $T_{1,a,2a+5}$, where $a \geq 2$. Without loss of generality, let $H' = \cup_i H'_i$. From (4.9) and (4.13), we can easily get that

$$\prod_{i=1} h(H'_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2}), \tag{4.14}$$

$$H'_i \in \mathcal{T}_1 \cup \{K_1\} \cup \{C_m, D_m, T_{1,b,c} | m \geq 4, 2 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}. \tag{4.15}$$

Now we consider the minimum real roots on both sides of (4.14), namely, $\beta(H')$ and $\beta(G)$. Assume that T_{1,s_1,s_2} is a component of G with $\beta(G) = \beta(T_{1,s_1,s_2})$. Clearly, $3 \leq s_1 \leq 10$ and $s_2 \geq s_1$. From (4.14), we see that H' must have a component, say H'_1, such that $\beta(H'_1) = \beta(T_{1,s_1,s_2})$. As $\beta(C_n) = \beta(T_{1,1,n-2})$ and $\beta(D_n) = \beta(T_{1,2,n-3})$ for $n \geq 4$, we know, by Theorem 3.2.6 and (4.15), that $H'_1 \in \{T_{1,s_1,s_2}, T_{1,a,a+1} | a \geq 3\}$. Suppose that $H'_1 \cong T_{1,a,a+1}$ and $T_{1,a,a+1} \ncong T_{1,s_1,s_2}$. From Theorem 3.2.6(iii) and $\beta(T_{1,a,a+1}) = \beta(T_{1,s_1,s_2})$, we have $T_{1,a-1,2a+3} \cong T_{1,s_1,s_2}$, which contradicts the fact that $s_2 \neq 2s_1 + 5$. Thus, $H'_1 \cong T_{1,s_1,s_2}$. Eliminating a factor $h(T_{1,s_1,s_2})$ from both sides of (4.14), we arrive at

$$\prod_{i=2} h(H'_i) = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \prod_{l_1,l_2} h(T_{1,l_1,l_2})/h(T_{1,s_1,s_2}). \tag{4.16}$$

From (4.16), we can obtain the following fact by repeating the above argument.

Fact 1. For each component T_{1,l_1,l_2} of G, there must be a component H'_i of H' such that $H'_i \cong T_{1,l_1,l_2}$.

Eliminating the factor $\prod_{l_1,l_2} h(T_{1,l_1,l_2})$ of $h(G)$ from both sides of (4.14), it follows immediately that

$$\prod_{i=1} h(H_i'') = \prod_i h(C_{n_i}) \prod_j h(D_{m_j}) \tag{4.17}$$

and

$$H_i'' \in \mathcal{T}_1 \cup \{K_1\} \cup \{C_m, D_m, T_{1,b,c} | m \geq 4, 2 \leq b \leq c, b \neq c, c \neq b+3, c \neq 2b+5\}. \tag{4.18}$$

Since $p((\cup_i C_{n_i}) \cup (\cup_j D_{m_j})) = q((\cup_i C_{n_i}) \cup (\cup_j D_{m_j}))$, we have $p(\cup_i H_i'') = q(\cup_i H_i'')$. So, from (4.18), we have

$$H_i'' \in \{C_n, D_m | n \geq 4, m \geq 4\}. \tag{4.19}$$

From the condition of the theorem and Lemma 3.2.6, we have $\beta(D_{m_j}) < -4 < \beta(C_n)$, for $m_j \geq 9$ and $n \geq 4$. From Corollary 3.2.1, we can get the following fact by comparing the minimum real roots on both sides of equation (4.17) .

Fact 2. For each component D_{m_j} of G, there must be a component H_i' such that $H_i' \cong D_{m_j}$ in H'.

Eliminating the factor $\prod_j h(D_{m_j})$ of $h(G)$ from both sides of (4.17), it follows that

$$\prod_i h(H_i''') = \prod_i h(C_{n_i}), \; H_i''' \in \{C_n, D_m | n \geq 4, m \geq 4\}. \tag{4.20}$$

The following fact is obtained from (4.18), the condition of the theorem and Lemma 4.2.5.

Fact 3. $\cup_i H_i''' \cong \cup_i C_{n_i}$.

From Facts 1, 2 and 3, it is clear that $H' \cong G$. Suppose that H has at least one component $T_{1,a,2a+5}$. Obviously, H' must contain components $T_{1,a+1,a+2}$ and C_{a+2}. Recalling that $H' \cong G$, we have that G must contain the components $T_{1,a+1,a+2}$ and C_{a+2}. This contradicts the condition of the theorem. So, H does not contain the component $T_{1,a,2a+5}$. Therefore, $H \cong H' \cong G$. This completes the proof of the sufficiency of the theorem.

From Lemma 3.2.8(iii), the necessity of the condition is obvious. □

From Theorems 4.4.1, 2.2.4 and 2.2.5, we have

Corollary 4.5.1. *Let* $G_i \in \{C_i | i \geq 5, i \not\equiv 2(mod\ 4)\} \cup \{D_j | j \geq 9, j \not\equiv 2(mod\ 3), j \not\equiv 3(mod\ 5)\} \cup \{T_{1,l_1,l_2} | 3 \leq l_1 \leq 6, l_1 \leq l_2, l_1 \neq l_2, l_1 \neq l_2 + 1, l_2 \neq 2l_1 + 5\}$ *and* $(l_1, l_2) \notin \{(3, 3k), (3, 4k-1), (4, 4k+1), (4, 5k-1), (4, 7k), (5, 3k+2), (5, 4k+4), (5, 5k+2), (6, 3k+3), (6, 7k-1) | k \geq 1\}$. *Then* $\overline{\cup_i G_i}$ *is* χ*–unique.*

Theorem 4.5.2. *Let* $3 \leq l_1 \leq 10$ *and* $l_1 \leq l_2$. *If* $h(P_m) \not| h(T_{1,l_1,l_2})$ *for any* $m \geq 2$, *then* $K_n - E(\cup_{l_1,l_2} T_{1,l_1,l_2})$ *is* χ*–unique if and only if* $l_2 \neq 2l_1 + 5$, *where* $n \geq \sum\limits_{l_1,l_2} |V(T_{1,l_1,l_2})|$.

Proof. Obviously, $\overline{K_n - E(\cup_{l_1,l_2} T_{1,l_1,l_2})} = rK_1 \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$, where $r = n - \sum\limits_{l_1,l_2} |V(T_{1,l_1,l_2})|$. Let $G = rK_1 \cup (\cup_{l_1,l_2} T_{1,l_1,l_2})$. Because of Theorem 1.1.1, we only consider the necessary and sufficient conditions for G to be adjointly unique.

Let H be a graph such that $h(H) = h(G)$. Suppose that $H = \cup_i H_i$, where each H_i is connected. By Theorem 2.1.1 and Lemmas 4.2.1 and 4.2.3, we have

$$\prod_i h(H_i) = x^r \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \tag{4.21}$$

and

$$\sum_i R_1(H_i) = \sum_{l_1,l_2} R_1(T_{1,l_1,l_2}) = 0. \tag{4.22}$$

With an argument similar to that of Theorem 4.5.1, by the condition of the theorem, Theorem 3.4.1 and Lemma 4.5.1, we have

$$H_i \in \{C_n, D_m, T_{1,b,c}, K_1 | n \geq 4, m \geq 4, 1 \leq b \leq c, b \neq c, c \neq b + 3\}. \tag{4.23}$$

We construct a graph H' from H by replacing each component $T_{1,a,2a+5}$ by two components C_{a+2} and $T_{1,a+1,a+2}$ until none of components is isomorphic to $T_{1,a,2a+5}$, where $a \geq 2$. Without loss of generality, let $H' = \cup_i H'_i$. By (4.21) and (4.23), we obtain that

$$\prod h(H'_i) = x^r \prod_{l_1,l_2} h(T_{1,l_1,l_2}) \tag{4.24}$$

and

$$H'_i \in \mathcal{T}_1 \cup \{K_1\} \cup \{C_m, D_m, T_{1,b,c} | m \geq 4, 2 \leq b \leq c, b \neq c, c \neq b + 3, c \neq 2b + 5\}. \tag{4.25}$$

Similar to the proof of Theorem 4.5.1, by comparing the minimum real roots on both sides of (4.24), we have

Fact 4. For each component T_{1,l_1,l_2} of G, there must be a component H'_i of H' such that $H'_i \cong T_{1,l_1,l_2}$.

Eliminating the factor $\prod_{l_1,l_2} h(T_{1,l_1,l_2})$ of $h(G)$ from both sides of (4.24), it follows immediately that

$$\prod h(H''_i) = x^r. \tag{4.26}$$

From Fact 4 and (4.26), we have $H' \cong G$. Assume that H has at least one component $T_{1,a,2a+5}$. Then H' must contain a component C_{a+2} and C_{a+2} is a component of G, contradicting (4.26). So, $H \cong H' \cong G$. The proof of the sufficiency of the condition is complete.

From (iii) of Lemma 3.2.8, the necessity of the condition is obvious. $\square$

Corollary 4.5.2. *Let* $G_i \in \{T_{1,l_1,l_2} | 3 \le l_1 \le 6, l_1 \le l_2, l_1 \ne l_2, l_1 \ne l_2 + 1, l_2 \ne 2l_1 + 5\}$ *and* $(l_1, l_2) \notin \{(3, 3k), (3, 4k-1), (4, 4k+1), (4, 5k-1), (4, 7k), (5, 3k+2), (5, 4k+4), (5, 5k+2), (6, 3k+3), (6, 7k-1) | k \ge 1\}$. *Then* $K_n - E(\cup_i G_i)$ *is* χ*–unique, where* $n \ge \sum\limits_{l_1,l_2} |V(T_{1,l_1,l_2})|$.

4.6 An invariant $R_2(G)$ for adjointly equivalent graphs and its application

In this section, we first give some new properties of the invariant $R_2(G)$. Then we find all adjointly equivalent graphs of two classes of graphs with $R_1(G) = -1$.

In [27], Dong, Teo, Little and Hendy introduced an invariant for adjointly equivalent graphs as follows.

Definition 4.6.1. *([27]) Let G be a graph. Then an invariant $R_2(G)$ of G is defined by*

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G).$$

It is clear that for any two graphs G and H, if $h(G) = h(H)$, then $R_2(G) = R_2(H)$. The following lemmas can be found in [27].

Lemma 4.6.1. *([27]) Let G be a graph with k components G_1, G_2, $\cdots$, G_k. Then*

$$R_2(G) = \sum_{i=1}^{k} R_2(G_i).$$

Lemma 4.6.2. *([27])*
(i) $R_2(P_1) = 0$, $R_2(P_2) = -1$ *and* $R_2(P_n) = -2$, *for* $n \geq 3$;
(ii) $R_2(K_3) = -2$ *and* $R_2(C_n) = 0$, *for* $n \geq 4$;
(iii) $R_2(T_{1,1,1}) = -1$ *and* $R_2(T_{1,1,n}) = 0$, *for* $n \geq 2$;
(iv) $R_2(D_4) = 0$ *and* $R_2(D_n) = 1$, *for* $n \geq 5$;
(v) $R_2(F_6) = 5$ *and* $R_2(F_n) = 4$, *for* $n \geq 7$;
(vi) $R_2(K_4^-) = 3$.

Lemma 4.6.3. *([80]) Let G be a connected graph with $R_1(G) = -1$ and $q(G) \geq p(G)$. Then $b_3(G) \geq b_3(A_{q(G)}) + k$, $k \geq 0$ and*
(i) $k = 0$ *if and only if* $G \in \{A_{q(G)} | q(G) \geq 5\} \cup \{B_{q(G)}, C_3(P_2, P_2), K_4^- | q(G) \geq 7\}$,
(ii) $b_3(B_6) = b_3(A_6) + 1$ *and* $b_3(B_7) = b_3(A_7) = b_3(F_6) - 2$,
(iii) $b_3(B_n) = b_3(A_n) = b_3(F_{n-1}) - 1$, *for* $n \geq 8$.

Theorem 4.6.1. *Let G be a connected graph with $R_1(G) = -1$ and $q(G) \geq p(G)$. Then $R_2(G) \geq 3$ and the equality holds if and only if $G \in \{A_{q(G)} | q(G) \geq 5\} \cup \{B_{q(G)}, C_3(P_2, P_2), K_4^- | q(G) \geq 7\}$. In particular, we have that $R_2(A_n) = 3$, for $n \geq 5$, $R_2(B_6) = 4$, and $R_2(B_n) = 3$, for $n \geq 7$.*

Proof. Suppose that G is a connected graph with $R_1(G) = -1$ and $q(G) \geq p(G) + 1$. By Lemma 4.2.3 (iii), we have that $G \in \{K_4^-, F_n | n \geq 6\}$. From Lemma 4.6.2, $R_2(F_n) > 3$, for $n \geq 6$, and $R_2(K_4^-) = 3$.

Suppose that G is a connected graph with $R_1(G) = -1$ and $q(G) = p(G)$. From Definitions of $R_1(G)$ and $R_2(G)$, it follows by calculation that

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)[(R_1(G) - b_1(G)] - b_1(G).$$

Since $b_1(G) = q(G) = q(F_{q(G)-1})$, we have

$$R_2(G) = R_2(F_{q(G)-1}) + b_3(G) - b_3(F_{q(G)-1}).$$

By Lemmas 4.6.2 and 4.6.3, we have that the theorem holds when $q(G) \geq 7$. If $q(G) = 5$ or $q(G) = 6$, then $G \in \{A_5, A_6, B_6\}$. It is easy to verify that $R_2(B_6) = 4$ and $R_2(A_5) = R_2(A_6) = 3$. □

From Lemma 4.2.4 and Definition 4.6.1, we have

Theorem 4.6.2. *Let G be a graph with p vertices and q edges. If the degree sequence of G is $(d_1, d_2, d_3, \cdots, d_p)$, then*

$$R_2(G) = \frac{4q}{3} - 2\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij \in E(G)} d_i d_j - \sum_{i \in M} M(i) d_i + 4N_A(G) + N(K_4).$$

From Theorem 4.6.2, the following lemma follows.

Lemma 4.6.4. *$R_2(T_{1,l_2,l_3}) = 1$, for all $2 \leq l_2 \leq l_3$, and $R_2(T_{l_1,l_2,l_3}) = 2$, for all $2 \leq l_1 \leq l_2 \leq l_3$.*

Theorem 4.6.3. *Let $G = A_n \cup (\cup_{i=1}^{k} C_{m_i})$, where $m_i \not\equiv 2(mod\ 4)$ and $m_i \geq 5$, for all i, and $n \not\equiv 2(mod\ 3)$ and $n \geq 5$. Then $[G]_h = \{G\}$ except for $[G]_h = \{A_7 \cup (\cup_{i=1}^{k} C_{m_i}), B_7 \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 7$; in particular, $\overline{G}$ is χ-unique if and only if $n \neq 7$.*

Proof. Let H be a graph such that $h(H) = h(G)$ and assume that $H = \cup_{i=1}^{s} H_i$. From Lemmas 4.2.1 and 4.2.3, we have

$$R_1(H) = R_1(G) = \sum_{i=1}^{s} R_1(H_i) = R_1(A_n) + \sum_{i=1}^{k} R_1(C_{m_i}) = -1. \qquad (4.27)$$

By the condition of the theorem and Theorem 2.2.5, it is not hard to see that $h(P_m) \nmid h(C_{m_i})$ and $h(P_m) \nmid h(A_n)$, for all $m \geq 2$. By Lemma 4.5.1, we have that $h(P_m) \nmid h(H)$, for all $m \geq 2$. As $h_1(C_3) = h_1(P_4)$, we know that H does not contain any P_m or C_3 as its component. So, from (4.27) and Lemmas 4.2.1 and 4.2.3, we have that there exists exactly one component in H, say H_1, such that $R_1(H_1) = -1$ and $R_1(H_i) = 0$, for all $i \geq 2$. Without loss of generality, from Lemma 4.2.3, we can assume that

$$H = rK_1 \cup H_1 \cup (\bigcup_{i} C_{n_i}) \cup fD_4 \cup (\bigcup_{j \in S} D_{u_j}) \cup lT_{1,1,1} \cup (\bigcup_{T \in \mathcal{T}} T), \qquad (4.28)$$

where H_1 is connected and $R_1(H_1) = -1$, $u_j \geq 5$, for all $j \in S$, $n_i \geq 4$, for all i and $\mathcal{T} = \{T_{l_1,l_2,l_3} | 1 \leq l_1 \leq l_2 \leq l_3 \text{ and } (l_1, l_2, l_3) \neq (1,1,1)\}$.

Let $q(H_1) = p(H_1) + t$. By Lemma 4.2.3, it is clear that $t \leq 1$ and $t = 1$ if and only if $H_1 \in \{F_v, K_4^-\}$. Since $h(P_2)|h(K_4^-)$, we must have $H_1 \cong F_v$, for $t = 1$. From (4.28), it follows that

$$\begin{aligned} q(H) &= p(H_1) + t + \sum_i p(C_{n_i}) + fp(D_4) + \sum_{j \in S} p(D_{u_j}) + \\ &\quad lp(T_{1,1,1}) - l + \sum_{T \in \mathcal{T}} p(T) - |\mathcal{T}| \\ &= p(H) - r + t - l - |\mathcal{T}|. \end{aligned} \tag{4.29}$$

Recalling that $p(G) = q(G)$, we have $p(H) = q(H)$. Thus, from (4.29),

$$r + l + |\mathcal{T}| = t. \tag{4.30}$$

From the fact that $t \leq 1$ and $t = 1$ if and only if $H_1 \cong F_v$ and $v \geq 6$, it suffices to consider the following cases.

Case 1. $t = 1$. So, from (4.30), $l \leq 1$. Obviously, $H_1 \cong F_v$, where $v \geq 6$.

From $h(H) = h(G)$, we have $R_2(G) = R_2(H)$. From Lemmas 4.6.1, 4.6.2 and 4.6.4 and Theorem 4.6.1, we have that $R_2(H) \geq R_2(F_v) + |S| - l$ and $R_2(G) = 3$. Thus, $R_2(H) = 3$ if and only if $l = 1$, $|S| = 0$ and $H_1 \cong F_v$, where $v \geq 7$. From (4.30), $r = |\mathcal{T}| = 0$. From (4.28), it follows immediately that

$$H = F_v \cup (\bigcup_i C_{n_i}) \cup fD_4 \cup T_{1,1,1}. \tag{4.31}$$

Since $h(G) = h(H)$ and $h(C_4) = h(D_4)$, from Theorem 2.1.1, the condition of theorem and (4.31), we have

$$h(A_n) \prod_{i=1}^{k} h(C_{m_i}) = h(F_v)h(T_{1,1,1})[h(C_4)]^f \prod_i h(C_{n_i}). \tag{4.32}$$

From (i) and (ii) of Theorem 3.2.5 and Lemma 3.2.6, we know that $\beta(G) = \beta(A_n)$ and $\beta(H) = \beta(F_v)$. From Lemma 3.2.7, it is clear that $h(P_4)|h(F_{17})$, $h(P_2)|h(F_{11})$, $h(P_4)|h(F_7)$ and $h(P_2)|h(F_8)$. Recalling that $h(P_m) \nmid h(H)$, for all $m \geq 2$, we have that $v \neq 7, 8, 11$ and 17. Thus, by the condition of the theorem and (v) of Theorem 3.2.5, we have that $n \neq 5, 6, 7, 8$.

Suppose that $n = 9$. Clearly $v = 9$. From (v) of Lemma 3.2.7 and (4.32), it follows that

$$h(B_6)h(T_{1,1,1})\prod_{i=1}^{k} h(C_{m_i}) = h(B_6)h(C_4)h(T_{1,1,1})[h(C_4)]^f \prod_i h(C_{n_i}).$$

So,

$$\prod_{i=1}^{k} h(C_{m_i}) = h(C_4)[h(C_4)]^f \prod_i h(C_{n_i}).$$

By Lemma 4.2.5, $\cup_{i=1}^{k} C_{m_i} \cong C_4 \cup_i C_{n_i} \cup fC_4$, which contradicts $m_i \geq 5$ for all i.

Suppose that $n \geq 10$. Obviously, $v \geq 10$. Again by Theorem 3.2.5, it is not hard to see that, for $n \geq 10$ and $v \geq 10$, $\beta(A_n) < \beta(F_v)$. This contradicts $h(G) = h(H)$.

Case 2. $t = 0$. From (4.30), $r = l = |\mathcal{T}| = 0$. So, from (4.28),

$$H = H_1 \cup (\bigcup_i C_{n_i}) \cup fD_4 \cup (\bigcup_{j \in S} D_{u_j}), \tag{4.33}$$

where H_1 is connected and $R_1(H_1) = -1$, $u_j \geq 5$, for all $j \in S$, and $n_i \geq 4$, for all i.

Since $R_2(G) = R_2(H)$, from Lemmas 4.6.1, 4.6.2 and 4.6.4 and Theorem 4.6.1 we have that $R_2(H) \geq R_2(H_1) + |S|$ and $R_2(G) = 3$. Thus, $R_2(H) = 3$ if and only if $|S| = 0$ and $H_1 \in \{A_v | v \geq 5\} \cup \{B_v, C_3(P_2, P_2), K_4^- | v \geq 7\}$. Note that, for all $m \geq 2$, $h(P_m) \not| h(H_1)$, whereas $h(P_2) | h(C_3(P_2, P_2))$, $h(P_2) | h(K_4^-)$ and $h(P_2) | h(A_5)$. So, $H_1 \in \{A_v | v \geq 6\} \cup \{B_v | v \geq 7\}$. From (4.33), it follows immediately that

$$H = H_1 \cup (\bigcup_i C_{n_i}) \cup fD_4. \tag{4.34}$$

Since $h(C_4) = h(D_4)$, it follows, from (4.34), that

$$h(A_n)\prod_{i=1}^{k} h(C_{m_i}) = h(H_1)[h(C_4)]^f \prod_i h(C_{n_i}). \tag{4.35}$$

Suppose that $H_1 \cong B_v$ and $v \geq 7$. From Theorem 3.2.5, Lemma 3.2.6 and (4.35), we have $\beta(G) = \beta(A_n)$ and $\beta(H) = \beta(B_v)$. So, $\beta(B_v) = \beta(A_n)$.

Hence, from Theorem 3.2.5, we have that $\beta(B_v) = \beta(A_n)$ if and only if $v = 10$ and $n = 6$, or $v = 7$ and $n = 7$.

If $n = 6$, then $v = 10$. Since $h(B_{10}) = h(C_4)h(A_6)$, it follows from (4.35) that

$$\prod_{i=1}^{k} h(C_{m_i}) = h(C_4)[h(C_4)]^f \prod_i h(C_{n_i}).$$

From Lemma 4.2.5, we have that $\cup_{i=1}^{k} C_{m_i} \cong C_4 \cup_i C_{n_i} \cup fC_4$. This again contradicts $m_i \geq 5$ for all i.

If $n = 7$, then $v = 7$. By Lemma 3.2.7, $A_7 \sim_h B_7$. Thus, we have that $f = 0$ and $[A_7 \cup (\cup_{i=1}^{k} C_{m_i})]_h = \{A_7 \cup (\cup_{i=1}^{k} C_{m_i}), B_7 \cup (\cup_{i=1}^{k} C_{m_i})\}$.

From the above arguments, we must have $H_1 \cong A_v$ if $v \geq 8$. Comparing the minimum real roots on both sides of (4.35), from Lemma 3.2.6 and Theorem 3.2.5, we have $H_1 \cong A_n$. Thus, it follows from (4.35) that

$$\prod_{i=1}^{k} h(C_{m_i}) = [h(C_4)]^f \prod_i h(C_{n_i}). \tag{4.36}$$

By Lemma 4.2.5 and (4.36), $f = 0$ and $H \cong G$ for $n \neq 7$ and $n \not\equiv 2(mod\ 3)$. This completes the proof of the theorem. $\square$

Corollary 4.6.1. *For $n \geq 5$ and $n \not\equiv 2(mod\ 3)$, $\overline{A_n}$ is χ-unique if and only if $n \neq 7$.*

Theorem 4.6.4. *Let $G = B_n \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})$, where $m_i \not\equiv 2(mod\ 4)$, and $m_i \geq 5$ and $m_i \neq 9, 15$, for all i, and $n \geq 7$. Then $[G]_h = \{G\}$ except for the following cases:*

(i) $[G]_h = \{G, A_7 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 7$;

(ii) $[G]_h = \{G, F_{13} \cup T_{1,1,1} \cup (a-1)C_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 8$ and $a \geq 1$;

(iii) $[G]_h = \{G, F_{15} \cup T_{1,1,1} \cup C_5 \cup aC_9 \cup (b-1)C_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 9$ and $b \geq 1$;

(iv) $[G]_h = \{G, A_6 \cup C_4 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i}), A_6 \cup D_4 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 10$.

In particular, $\overline{G}$ is χ-unique if and only if $n \neq 7, 10$, and $a = 0$ when $n = 8$, and $b = 0$ when $n = 9$.

Proof. Let H be a graph such that $h(H) = h(G)$, and assume that $H = \cup_{i=1}^{s} H_i$. From Lemmas 4.2.1 and 4.2.3, we have

$$R_1(H) = \sum_{i=1}^{s} R_1(H_i) = R_1(B_n) + aR_1(C_9) + bR_1(C_{15}) + \sum_{i=1}^{k} R_1(C_{m_i}) = -1. \tag{4.37}$$

With the same argument as Theorem 4.6.3, we can assume, from (4.37) and Lemmas 4.2.1 and 4.2.3, that

$$H = rK_1 \cup H_1 \cup (\bigcup_i C_{n_i}) \cup fD_4 \cup (\bigcup_{j\in S} D_{u_j}) \cup lT_{1,1,1} \cup (\bigcup_{T\in\mathcal{T}} T), \tag{4.38}$$

where H_1 is connected and $R_1(H_1) = -1$, $u_j \geq 5$, for all $j \in S$, $n_i \geq 4$, for all i, and $\mathcal{T} = \{T_{l_1,l_2,l_3} | 1 \leq l_1 \leq l_2 \leq l_3 \text{ and } (l_1, l_2, l_3) \neq (1, 1, 1)\}$.

Let $q(H_1) = p(H_1) + t$. With an argument similar to that of Theorem 4.6.3, we can obtain that

$$t \leq 1 \text{ and } t = 1 \text{ if and only if } H_1 \cong F_v,\ v \geq 6, \tag{4.39}$$

and

$$r + l + |\mathcal{T}| = t. \tag{4.40}$$

So, we consider the following cases.

Case 1. $t = 1$. So, $l \leq 1$. Obviously, $H_1 \cong F_v$ and $v \geq 6$.

From $h(H) = h(G)$, we have $R_2(G) = R_2(H)$. Using an argument similar to that of Case 1 of Theorem 4.6.3, from Lemmas 3.6.1, 3.6.2 and 3.6.4 and Theorem 3.2.5 we can show that $l = 1$, $|S| = r = |\mathcal{T}| = 0$ and $H_1 \cong F_v$, where $v \geq 7$. So, it follows immediately, from (4.38) and $h(D_4) = h(C_4)$, that

$$H = F_v \cup (\bigcup_i C_{n_i}) \cup fD_4 \cup T_{1,1,1} \tag{4.41}$$

and

$$h(B_n)[h(C_9)]^a[h(C_{15})]^b \prod_{i=1}^{k} h(C_{m_i}) = h(F_v)h(T_{1,1,1})[h(C_4)]^f \prod_i h(C_{n_i}). \tag{4.42}$$

From Theorem 3.2.5 and Lemma 3.2.6, we know that $\beta(G) = \beta(B_n)$ and $\beta(H) = \beta(F_v)$. From $\beta(G) = \beta(H)$, again by Theorem 3.2.5, it is not hard to see that there exists a positive integer j such that $v = 2j+1$ and $n = j+2$, where $j \geq 5$. Thus, from (4.42), we have that

$$h(B_{j+2})[h(C_9)]^a[h(C_{15})]^b \prod_{i=1}^{k} h(C_{m_i}) = h(F_{2j+1})h(T_{1,1,1})[h(C_4)]^f \prod_{i} h(C_{n_i}). \tag{4.43}$$

Since $h(F_{2j+1})h(K_1) = h(D_j)h(B_{j+2})$ for $j \geq 4$, it follows from (4.43) that

$$h(K_1)[h(C_9)]^a[h(C_{15})]^b \prod_{i=1}^{k} h(C_{m_i}) = h(D_j)h(T_{1,1,1})[h(C_4)]^f \prod_{i} h(C_{n_i}). \tag{4.44}$$

Now we consider the minimum real roots of both sides of equation (4.44). We denote respectively by $\beta(right)$ and $\beta(left)$ the minimum real roots of the right-hand side and of the left-hand side of equation (4.44).

Suppose that there are i and j such that $\beta(right) = \beta(C_{m_i})$ and $\beta(left) = \beta(C_{n_j})$. By Corollary 3.2.1, $C_{m_i} \cong C_{n_j}$. After eliminating a factor $h(C_{m_i})(= h(C_{n_j}))$ from both sides of equality (4.44), we repeat the elimination procedure and must obtain that there are i' and j' such that $\beta(right) = \beta(C_{m_{i'}})$ and $\beta(left) = \beta(D_{j'})$. Clearly, $\beta(C_{m_{i'}}) = \beta(D_{j'})$. By Corollary 3.2.1 and Lemma 4.3.2, we have that $m_{i'} = 9$ and $j' = 6$, or $m_{i'} = 15$ and $j' = 7$, or $m_{i'} = 6$ and $j' = 5$. From the condition of the theorem, $m_j \neq 6$, for all i. Note that $h(B_9)h(C_{15}) = h(F_{15})h(T_{1,1,1})h(C_5)$ and $h(B_8)h(C_9) = h(F_{13})h(T_{1,1,1})$. So, it is not hard to see that (4.43) holds if and only if $a \geq 1$ and $j = 6$, or $b \geq 1$ and $j = 7$. By (4.42), we have that

$$[h(C_9)]^{a-1}[h(C_{15})]^b \prod_{i=1}^{k} h(C_{m_i}) = [h(C_4)]^f \prod_{i} h(C_{n_i})$$

or

$$h(C_5)[h(C_9)]^a[h(C_{15})]^{b-1} \prod_{i=1}^{k} h(C_{m_i}) = [h(C_4)]^f \prod_{i} h(C_{n_i}).$$

By Lemma 4.2.5, we know that, if $t = 1$, then $f = 0$. Thus, $H \in \{B_8 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i}), F_{13} \cup T_{1,1,1} \cup (a-1)C_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 8$, and $H \in \{B_9 \cup aC_9 \cup bC_{15} \cup (\cup_{i=1}^{k} C_{m_i}), F_{15} \cup T_{1,1,1} \cup C_5 \cup aC_9 \cup (b-1)C_{15} \cup (\cup_{i=1}^{k} C_{m_i})\}$, for $n = 9$. This proves (ii) and (iii).

Case 2. $t = 0$. So, $r = l = |\mathcal{T}| = 0$.

With a proof similar to that of Case 2 of Theorem 4.6.3, we can obtain that

$$h(B_n)\prod_{i=1}^{k} h(C_{m_i}) = h(H_1)[h(C_4)]^f \prod_i h(C_{n_i}) \tag{4.45}$$

and $H_1 \in \{A_v|v \geq 6\} \cup \{B_v|v \geq 7\}$.

From Theorem 3.2.5 and Lemma 3.2.6, $\beta(G) = \beta(B_n)$ and $\beta(H) = \beta(A_v)$. Again by Theorem 3.2.5, it is not difficult to see that $\beta(B_n) = \beta(A_v)$ if and only if $n = 7$ and $v = 7$, or $n = 10$ and $v = 6$.

Suppose that $n = 7$ and $v = 7$. By Lemma 3.2.7, $B_7 \sim_h A_7$. By Lemma 4.2.5 and (4.45), we have that $f = 0$ and (i) of the theorem holds.

Suppose that $n = 10$ and $v = 6$. By Lemma 3.2.7, $B_{10} \sim_h A_6 \cup C_4$ and $B_{10} \sim_h A_6 \cup D_4$. By (4.45), we have

$$h(C_4)\prod_{i=1}^{k} h(C_{m_i}) = [h(C_4)]^f \prod_i h(C_{n_i}).$$

Comparing the minimum real roots of both sides of the above equation, by $D_4 \sim_h C_4$ we know that (iv) of the theorem is true.

Since $n \neq 5, 6$, we have that $n \geq 8$ and $v \geq 8$. Comparing the minimum real roots of both sides of (4.45), from Lemma 3.2.6 and Theorem 3.2.5 we have $H_1 \cong B_n$. Thus, it follows from (4.45) that

$$\prod_{i=1}^{k} h(C_{m_i}) = [h(C_4)]^f \prod_i h(C_{n_i}). \tag{4.46}$$

By Lemma 4.2.5 and (4.46), we have that $f = 0$ and $H \cong G$. □

By Lemma 3.2.7 (vi) and Theorem 4.6.4, we have

Corollary 4.6.2. *For all $n \geq 6$, $\overline{B_n}$ is χ-unique if and only if $n \neq 6, 7$ and 10.*

4.7 A new invariant $R_3(G)$ for adjointly equivalent graphs and its application

We introduce a new invariant $R_3(G)$ in this section and give some useful properties. By using these properties, we get some new chromatically unique graphs.

For any graph G, set $R_3(G) = R_1(G)+q(G)-p(G)$. Clearly, $R_3(K_1) = -1$, $R_3(P_2) = R_3(P_3) = 0$, and $R_3(K_3) = 1$. It is not hard to see that $R_3(G)$ is an invariant. From Lemmas 4.2.1 to 4.2.3, we prove easily the following theorems.

Theorem 4.7.1. *Let G and H be two graphs such that $h(G, x) = h(H, x)$. Then*

$$R_3(G) = R_3(H).$$

Theorem 4.7.2. *Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then*

$$R_3(G) = \sum_{i=1}^{k} R_3(G_i).$$

Theorem 4.7.3. *Let G be a connected graph with $e = uv \in E(G)$. Then*

$$R_3(G) = R_3(G - e) - d_G(e) + 2,$$

where $d_G(e) = |N_G(u) \cup N_G(v)| - 2$.

Let $\mathcal{L} = \{P_n, C_{n+2}, D_{n+2}, F_{n+4}, K_4, K_4^-, K_3 | n \geq 2\}$. We have

Theorem 4.7.4. *Let G be a connected graph. Then*
(i) $R_3(G) \leq 1$, and the equality holds if and only if $G \cong K_3$.
(ii) $R_3(G) = 0$ if and only if $G \in \mathcal{L} \backslash \{K_3\}$.

Proof. (i) By induction on $q(G)$. By $R_3(K_1) = -1$ and $R_3(P_2) = R_3(P_3) = 0$, (i) holds for $q(G) \leq 2$.

Let G be a connected graph with $uv \in E(G)$. We denote by $(G - uv)_E$ the subgraph induced by $E(G - uv)$. Suppose $q(G) = k \geq 3$ and (i) holds for $q(G) < k$. Choose an edge $e \in E(G)$ such that $(G - e)_E$ is connected. Since $q(G) \geq 3$, one sees that $d_G(e) \geq 1$. We distinguish the following three cases:

Case 1. $d_G(e) \geq 2$ and $G - e$ is connected. By Theorem 4.7.3 and the induction hypothesis,

$$R_3(G) = R_3(G-e) - d_G(e) + 2 \leq R_3(G-e) \leq 1.$$

Note that $R_3(G) = 1$ if and only if $R_3(G-e) = 1$ and $d_G(e) = 2$. By the induction hypothesis, $G - e \cong K_3$, which contradicts the fact that G is a simple graph. So, $R_3(G) \leq 0$.

Case 2. $d_G(e) = 1$ and $G - e$ is connected. Note that $e = uv$ must be an edge of a triangle in G such that $d(u) = d(v) = 2$. If $G \cong K_3$, then $R_3(K_3) = 1$. If $G \ncong K_3$, then we can choose a new edge $u'v'$ in the triangle such that $d(u') \geq 3$ or $d(v') \geq 3$. It is obvious that $d_G(u'v') \geq 2$ and that $G - u'v'$ is connected. By Case 1, (i) holds.

Case 3. $G - e$ is disconnected. Remembering that $(G-e)_E$ is connected, we know that e must be a pendant edge, that is $G - e \cong (G-e)_E \cup K_1$. By $R_3(K_1) = -1$ and Theorems 4.7.2 and 4.7.3, $R_3(G) = R_3((G-e)_E) - d_G(e) + 1$. By the induction hypothesis, we have

$$R_3(G) = R_3((G-e)_E) - d_G(e) + 1 \leq R_3((G-e)_E) \leq 1,$$

and $R_3(G) = 1$ if and only if $R_3(G-e)_E = 1$ and $d_G(e) = 1$. So, by the induction hypothesis, $(G-e)_E \cong K_3$. Clearly, there exists no edge e in G such that $d_G(e) = 1$ and $(G-e)_E \cong K_3$. Hence $R_3(G) \leq 0$. This completes the proof of (i).

(ii) We prove (ii) by induction on $q(G)$. Since $R_3(K_1) = -1$ and $R_3(P_2) = R_3(P_3) = 0$, we know that (ii) holds if $q(G) \leq 2$.

Let G be a connected graph. Suppose $q(G) = k \geq 3$ and (ii) holds when $q(G) < k$. We can choose $e \in E(G)$ such that $(G-e)_E$ is connected and $d_G(e) \geq 1$. We distinguish the following two cases:

Case 1. $G - e$ is connected. Clearly, G contains at least one cycle. Note that $q(G) \geq 3$ and $R_3(K_3) = 1$. If $G \ncong K_3$, then we can choose an edge e such that $d_G(e) \geq 2$ and $G - e$ is connected. Since $R_3(G) = 0$, we have $d_G(e) = R_3(G-e) + 2$, by Theorem 4.7.3. So, for each $G \ncong K_3$, we only need consider the following two subcases:

Case 1.1. $R_3(G-e)=1$ and $d_G(e)=3$. By the induction hypothesis, $G-e\cong K_3$. Clearly, this is impossible.

Case 1.2. $R_3(G-e)=0$ and $d_G(e)=2$. By the induction hypothesis, $G-e\in\mathcal{L}\backslash\{K_3\}$. If $G-e\in\{P_n|n\geq 2\}$, then $G\in\{C_n,D_m|n\geq 4,m\geq 4\}$; if $G-e\in\{C_n|n\geq 4\}$, then $G\cong K_4^-$(only if $G-e\cong C_4$); if $G-e\in\{D_n|n\geq 4\}$, then $G\in\{K_4^-,F_n|n\geq 6\}$; if $G-e\cong K_4^-$, then $G\cong K_4$; if $G-e\in\{F_n,K_4|n\geq 6\}$, then G has no edge e such that $d_G(e)=2$, which is impossible.

Case 2. $G-e$ is disconnected. It is obvious that e must be a pendant edge. From Theorem 4.7.2, we have $R_3(G-e)=R_3((G-e)_E)-1$. Hence, we have $d_G(e)=R_3((G-e)_E)+1$ by Theorem 4.7.3. Since $R_3(G)\leq R_3((G-e)_E)$, we only consider the following two subcases:

Case 2.1. $R_3((G-e)_E)=1$ and $d_G(e)=2$. By the induction hypothesis, $(G-e)_E\cong K_3$. Since e must be a pendant edge in G, we have $G\cong D_4$.

Case 2.2. $R_3((G-e)_E)=0$ and $d_G(e)=1$. By the induction hypothesis, $(G-e)_E\in\mathcal{L}\backslash\{K_3\}$. If $(G-e)_E\in\{P_n|n\geq 2\}$, then $G\in\{P_n|n\geq 3\}$; if $(G-e)_E\in\{D_n|n\geq 4\}$, then $G\in\{D_n|n\geq 5\}$; if $(G-e)_E\in\{C_m,F_n,K_4,K_4^-|m\geq 4,n\geq 6\}$, then G has no an edge e such that $d_G(e)=1$, which is impossible.

Conversely, the necessity of (ii) follows from Lemma 4.2.3 immediately. $\Box$

Remark 4.7.1. *An alternative proof of Theorem 4.7.4 was given by Dong, Teo, Little and Hendy in [27].*

Theorem 4.7.5. *Let $\mathcal{F}_a=\{aK_3\cup\bigcup_i G_i|G_i\in\mathcal{L}$ and $h(K_3)\nmid h(G_i)\}$, where a is a nonnegative integer. Then $\mathcal{F}_a$ is adjointly closed.*

Proof: Suppose that $G\in\mathcal{F}_a$ and $H\sim_h G$. It is sufficient to prove that $H\in\mathcal{F}_a$. Namely, we show that H contains exactly a components K_3 and that each component of H belongs to $\mathcal{L}$.

Clearly, $h(H)=h(G)$. Denote by N_A the number of components K_3 in H. By Theorems 4.7.1, 4.7.2 and 4.7.4, we have $R_3(G)=R_3(H)=a$ and $N_A\geq a$. Since $h_1(K_3)$ is irreducible over the rational number field and $[h(K_3)]^{a+1}\nmid h(G)$, we have $[h_1(K_3)]^{a+1}\nmid h(H)$, so, $N_A\leq a$. Thus $N_A=a$, which implies H has exactly a components K_3 and $R_3(H_i)=0$ for every component H_i of H, except for K_3. By Theorem 4.7.4, $H_i\in\mathcal{L}$ for each i and

$H \in \mathcal{F}_a$. $\square$

We denote by $A, A_i, B, B_i, C, M_i, E$ and E_i the multisets of certain positive integers, where $i = 1, 2$.

Theorem 4.7.6. *Let a, t, r be nonnegative integers and let $G = (\cup_{i\in A} P_i) \cup (\cup_{j\in B} C_j) \cup (\cup_{k\in M} D_k) \cup (\cup_{s\in E} F_s) \cup aK_3 \cup tK_4^- \cup rK_4$, where $i \geq 2$, $i \not\equiv 4(mod\ 5)$ and i is even, $j \geq 5$, $k \not\equiv 3(mod\ 5)$ and $k \geq 9, s \not\equiv 2(mod\ 5)$ and $s \geq 6$. Then $\overline{G}$ is χ–unique if and only if $j \neq i+1$ if $2 \notin A$, or $j \neq 6, 9, 15$ and $j \neq i+1$ if $2 \in A$.*

Proof. By Theorem 1.1.1, it is not difficult to see that we need only consider the necessary and sufficient condition for G to be adjointly unique.

Let H be a graph such that $h(H) = h(G)$. By Theorem 2.2.5, we get that $h_1(K_3) = h_1(P_4) \nmid h_1(Y)$ for each $Y \in \{P_i | i \geq 2, i \not\equiv 4(mod\ 5)\} \cup \{C_j | j \geq 4\} \cup \{D_k | k \geq 4, k \not\equiv 3(mod\ 5)\} \cup \{F_s | s \geq 6, s \not\equiv 2(mod\ 5)\}$. So, by Theorem 4.7.5, $H \in \mathcal{F}_a$. Assume $H = aK_3 \cup H_1$ and $G = aK_3 \cup G_1$. Then, $h(G_1) = h(H_1)$. Without loss of generality, by Theorems 4.7.4 and 4.7.5, we may assume that

$$G_1 = (\cup_{i\in A} P_i) \cup (\cup_{j\in B} C_j) \cup (\cup_{k\in M} D_k) \cup (\cup_{s\in E} F_s) \cup tK_4^- \cup rK_4$$

and

$$H_1 = (\cup_{i_1\in A_1} P_{i_1}) \cup (\cup_{j_1\in B_1} C_{j_1}) \cup (\cup_{k_1\in M_1} D_{k_1}) \cup (\cup_{s_1\in E_1} F_{s_1}) \cup t_1K_4^- \cup r_1K_4,$$

where i, j, k and s satisfy the condition of the theorem.

It suffices to prove $H_1 \cong G_1$. Since $h_1(D_8) = (x^2+5x+4)h_1(K_3)$, we have that H_1 does not contain the component D_8. By calculation, we have that $h_1(K_4) = x^3 + 6x^2 + 7x + 1$ and $h(F_6) = x^4 + 7x^3 + 13x^2 + 7x + 1$. It is easy to verify that $\beta(K_4) < \beta(F_6)$. Thus, by Theorem 3.2.5 we have

$$\begin{aligned} &\beta(K_4) < \beta(F_6) < \beta(F_7) < \cdots < \beta(F_{n-1}) < \beta(F_n) < \beta(D_m) \\ &< \beta(D_{m-1}) < \cdots < \beta(D_9) < \beta(D_8) = \beta(K_4^-) = -4. \end{aligned}$$

By comparing the minimum real roots of $h_1(G_1)$ with those of $h_1(H_1)$, we know that $r = r_1, t = t_1, |E| = |E_1|$ and $M \subseteq M_1$. Eliminating all components G^* with $\beta(G^*) \leq -4$ from G_1 and H_1, we obtain

$$h((\cup_{i\in A} P_i) \cup (\cup_{j\in B} C_j)) = h((\cup_{i\in A_1} P_i) \cup (\cup_{j\in B_1} C_j) \cup (\cup_{k\in M_2} D_k)),$$

where $M_2 = M_1 \backslash M$.

By Theorem 4.3.2, we know that $\overline{(\cup_{i\in A} P_i) \cup (\cup_{j\in B} C_j)}$ is χ–unique if and only if $j \neq i+1$ if $2 \notin A$, or $j \neq 6, 9, 15$ and $j \neq i+1$ if $2 \in A$, when i and j satisfy the condition of the theorem. Hence $M_2 = \phi$ and $M = M_1$, which implies $H_1 \cong G_1$ and $H \cong G$. $\square$

In particular, from Theorem 4.7.6 we have

Corollary 4.7.1. *Let $k \not\equiv 3(mod\ 5)$ and $k \geq 9$, $s \not\equiv 2(mod\ 5)$ and $s \geq 6$. Then $\overline{(\cup_s F_s) \cup (\cup_k D_k)}$ is χ–unique.*

Corollary 4.7.2. *Let $K(n_1, n_2, \cdots, n_t)$ be the complete t-partite graph. Then $K(2, \cdots, 2,\ 3, \cdots, 3, 4, \cdots, 4)$ is χ-unique.*

Remarks

In this chapter, we investigated the chromaticity of some dense graphs, by using results obtained in Chapters 2 and 3. By using the fact $\beta(H) = \beta(G)$ if $H \sim_h G$ and Theorem 3.3.2, we established a necessary and sufficient condition of chromatic uniqueness of a dense graph such that its minimum degree is greater than or equal to the number of vertices minus 3, see Theorems 4.3.1, 4.3.2 and 4.3.3. Theorem 4.3.4 gave a necessary and sufficient condition for $\overline{\cup_i U_{n_i}}$, where $n_i \geq 6$, to be chromatically unique. Theorems 4.4.3 and 4.4.4 gave a necessary and sufficient condition for two graphs H and G with the minimum real roots of their adjoint polynomials greater than or equal to -4 to be adjointly equivalent. Theorem 4.3.1 gave a negative answer to two conjectures proposed in 2002 by Dong, Teo, Little and Hendy in Discrete Mathematics. By using results on invariants $R_1(G)$ and $R_2(G)$, some results on the divisibility between $h(P_n)$ with the adjoint polynomials of some special graphs, see Theorems 2.2.4 and 2.2.5, and some inequalities on the minimum real roots of the adjoint polynomials of some special graphs, see Theorems 3.2.5 and 3.2.6, we got some new chromatically unique graphs, see Theorems 4.5.1 and 4.5.2 and Corollaries 4.5.1 and 4.5.2 in Section 4.5, and all adjointly equivalent graphs for two classes of graphs with $R_1(G) = -1$, see Theorems

4.6.3 and 4.6.4 in Section 4.6. A new invariant $R_3(G)$ in Section 4.7 was introduced and some properties of $R_3(G)$ were found, see Theorems 4.7.1 to 4.7.4. By using these properties, we obtained some new chromatically unique graphs, see Theorem 4.7.6 and Corollaries 4.7.1 and 4.7.2. Although all graphs studied in this chapter have no close relation with those in Chapters 5 and 6, from a byproduct of this chapter, i.e., Corollary 4.7.2, we feel that we shall get some new results on the chromaticity of complete multipartite graphs, by using some properties of adjoint polynomials. In Chapter 5, we shall study the chromaticity of complete multipartite graphs, by using the results on the minimum real roots of adjoint polynomials.

Chapter 5

The Chromaticity of Complete Multipartite Graphs

5.1 Introduction

In Chapter 4, by using results obtained in Chapters 2 and 3 we investigated the chromaticity of dense graphs of form $K_n - E(H)$, where H is the union of some sparse graphs. Naturally, we wish to obtain some results on the chromaticity of complete multipartite graphs by applying some results in Chapters 2 and 3. Indeed, in this chapter, we get two interesting results, which confirm a conjecture and a problem posed by Koh and Teo in 1990, by applying some results on the minimum real roots of adjoint polynomials.

For the chromaticity of complete multipartite graphs, some researchers, including Chao, Dong, Koh, Novacky, Peng, Teo, Solzberg, López, Guidici *et al.*, focused on the chromaticity of complete bipartite graphs. After Teo and Koh [71] showed that all complete bipartite graphs $K(n, m)$ are χ-unique, for $n \geq m \geq 2$, many results on the chromaticity of complete multipartite graphs were found, see [18, 41, 43, 48, 83-88].

In particular, the authors in [7, 18, 43] obtained the following χ-unique graphs:

(i) $K(n_1, n_2, \cdots, n_t)$, for $|n_i - n_j| \leq 1$ and $n_i \geq 2, i = 1, 2, \cdots, t$;

(ii) $K(n, n, n + k)$, for $n \geq 2$ and $0 \leq k \leq 3$;

(iii) $K(n - k, n, n)$, for $n \geq k + 2$ and $0 \leq k \leq 3$;

(iv) $K(n - k, n, n + k)$, for $n \geq 5$ and $0 \leq k \leq 2$.

From 1998 to 2002, Zou and Shi in [83-88] gave some χ-unique complete tripartite graphs as follows.

(i) $K(n - k, n, n)$, for $n > k + k^2/3$;

(ii) $K(n, n, n + k)$, for $n > (k + k^2)/3$;

(iii) $K(n - k, n, n + k)$, for $n > k^2 + \frac{2\sqrt{3}}{3}k$;

(iv) $K(n - 4, n, n)$, for $n \geq 6$.

In [48], Li and Liu showed that $K(1, n_2, \cdots, n_t)$ is χ-unique if and only if $max\{n_i | i = 2, 3, \cdots, t\} \leq 2$.

In [41], Giudici and Lopez proved that the complete t-partite graph $K(n-1, n, \cdots, n, n+1)$, for $t \geq 2$ and $n \geq 3$, is χ-unique.

In 1990, the authors in [18, 43] proposed the following conjecture and problem:

Conjecture 5.1.1. *([18, 43]) For any integers n and k with $n \geq k + 2 \geq 4$, $K(n - k, n, n)$ is χ-unique.*

Problem 5.1.1. *([43]) Let $t \geq 2$. Is the graph $K(n_1, n_2, \cdots, n_t)$ χ-unique if $|n_i - n_j| \leq 2$, for all $i, j = 1, 2, \cdots, t$, and sufficiently large $min\{n_1, n_2, \cdots, n_t\}$?*

In this chapter, we first give some basic lemmas in Section 5.2. Then in Section 5.3 we investigate the chromaticity of $K(r, m, n)$, for $n \geq m \geq r \geq 2$, and give a positive answer to the above conjecture. We also show that $K(n-k, n-1, n)$, for $n \geq 2k$ and $k \geq 2$, is χ-unique. In Sections 5.4 and 5.5, by using some results of the minimum real roots of adjoint polynomials, we show that the complete t-partite graphs $K(n - k, n, n, \cdots, n)$ is χ-unique, for all $k \geq 2$, $n \geq k+2$ and $t \geq 3$. Some sufficient conditions for $K(n_1, n_2, \cdots, n_t)$ to be chromatically unique are established. Furthermore we solve Problem 5.1.1 completely.

For a graph G, let $S \in E(G)$ and $|S| = s$, we denote by $G - S$ (or $G - s$) the graph obtained from G by deleting all edges of S and write $\alpha_t(G - S) =$

$\alpha(G-S,t)-\alpha(G,t)$. Denote by $T_{n,t}$ the unique complete t-partite graph such that $n=\sum_{i=1}^{t} n_i$ and $|n_i-n_j|\le 1$, for all $i,j=1,2,\cdots,t$.

5.2 Some basic lemmas

For disjoint graphs G and H, we recall that $G+H$ denotes the join graph of G and H with vertex set $V(G)\cup V(H)$ and with edge set $\{xy|x\in V(G)$ and $y\in V(H)\}\cup E(G)\cup E(H)$. In fact, $\overline{G+H}=\overline{G}\cup\overline{H}$.

Lemma 5.2.1. *([4, 5]) Let G and H be two disjoint graphs. Then*

$$\sigma(G+H,x)=\sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1,n_2,n_3,\cdots,n_t),x)=\prod_{i=1}^{t}\sigma(O_{n_i},x).$$

Remark 5.2.1. *The above lemma is equivalent to the following: Let G and H be two disjoint graphs. Then*

$$h(\overline{G}\cup\overline{H},x)=h(\overline{G},x)h(\overline{H},x).$$

In particular,

$$h(\overline{K(n_1,n_2,n_3,\cdots,n_t)},x)=\prod_{i=1}^{t}h(K_{n_i},x).$$

Lemma 5.2.2. *([4, 70]) Let $S(n,k)$ denote the Stirling number of the second kind. Then*
(i) $\sigma(O_n,x)=\sum_{i=1}^{n}S(n,i)x^i$, where $O_n=\overline{K_n}$;
(ii) $S(n,1)=1$ and $S(n,2)=2^{n-1}-1$.

From Lemmas 5.2.1 and 5.2.2, we have

Lemma 5.2.3. *Let $G=K(n_1,n_2,\cdots,n_t)$. Then*
(i) for $1\le r\le t-1$, $\alpha(G,r)=0$,
(ii) $\alpha(G,t)=1$ and $\alpha(G,t+1)=\sum_{i=1}^{t}2^{n_i-1}-t$.

Lemma 5.2.4. *([6]) Let $G = K(n_1, n_2, \cdots, n_t)$ with n vertices. Then*
(i) $q(G) \leq q(T_{n,t})$, where equality holds if and only if $G = T_{n,t}$;
(ii) $q(T_{n,t}) - q(G) \geq max\{n_i | i = 1, \cdots, t\} - min\{n_i | i = 1, \cdots, t\} - 1$.

Lemma 5.2.5. *([43]) Let G and H be two graphs with $G \sim H$. Then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $N_A(G) = N_A(H)$ and $\alpha(G, r) = \alpha(H, r)$ for $r = 1, 2, 3, \cdots, p(G)$.*

Lemma 5.2.6. *Let $G = K(n_1, n_2, \cdots, n_t)$ with $\sum_{i=1}^{t} n_i = n$ and $n_1 \leq n_2 \leq \cdots \leq n_t$. Suppose that H is a graph such that $H \sim G$. Then there is a graph $F = K(m_1, m_2, \cdots, m_t)$ with $m_1 \leq m_2 \cdots \leq m_t$ and there is a set S of s edges in F such that $H = F - S$, $s = q(F) - q(G) \geq 0$ and F and G must satisfy the following conditions:*
(i) $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$,
(ii) $m_1 \geq \frac{n - \sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2}}{t}$, and
(iii) $n_1 \geq \frac{n - \sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2}}{t}$.

Proof. Since $H \sim G = K(n_1, n_2, \cdots, n_t)$, we have that $\sigma(H, x) = \sigma(G, x) = \sigma(K(n_1,\ n_2,\ \cdots, n_t), x)$. From Lemma 5.2.3, $\alpha(H, t) = \alpha(G, t) = 1$, which means that $V(H)$ has the unique t-independent partition, say $\{A_1, A_2, \cdots, A_t\}$. Since $\alpha(H, r) = \alpha(G, r) = 0$, for $r \leq t - 1$, we have that H is a t-partite graph. Let $|A_i| = m_i$, $i = 1, 2, \cdots, t$. Then there is a set S of s edges in $F = K(m_1, m_2, \cdots, m_t)$ such that $H = F - S$. Remembering that $\sigma(H, x) = \sigma(G, x)$, we have that $p(H) = p(G)$ and $q(H) = q(G)$. Clearly, $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$ and $s = q(F) - q(G) \geq 0$, which implies that (i) is true.

Now we prove (ii) and (iii). Let z denote the minimum value of m_1 such that $s \geq 0$. Then $q(K(z, m_2, m_3, \cdots, m_t)) - q(G) \geq 0$ for some $(m_2, m_3, \cdots, m_t)$. Denote by $K(z, y_2, \cdots, y_t)$ the complete t-partite graphs with $z \leq y_2 \leq y_3 \cdots \leq y_t$ and $|y_i - y_j| \leq 1$, for $i, j = 2, 3, \cdots, t$, where $\sum_{i=2}^{t} y_i = n - z$. Note that

$$q(K(m_1, \cdots, m_{i-1}, m_i + 1, m_{i+1}, \cdots, m_{j-1}, m_j - 1, m_{j+1}, \cdots m_t)) -$$

$$q(K(m_1,\cdots,m_{i-1},m_i,m_{i+1},\cdots,m_{j-1},m_j,m_{j+1},\cdots m_t))=m_j-m_i-1$$

for $i<j$ and $m_i<m_j$. So, it is not difficult to see that $q(K(z,y_2,\cdots,y_t))\geq q(K(z,m_2,m_3,\cdots,m_t))$, for all $(m_2,m_3,\cdots,m_t)$, and $q(K(z,y_2,\cdots,y_t))\leq z(n-z)+\frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2$. Therefore, one can see that if $s\geq 0$, then z must satisfy the following inequality:

$$z(n-z)+\frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2-q(G)\geq 0.$$

By solving the above inequality, we have

$$\frac{n-\sqrt{(t-1)((t-1)n^2-2q(G)t)}}{t}\leq z\leq\frac{n+\sqrt{(t-1)((t-1)n^2-2q(G)t)}}{t}.$$

Since $q(G)=\sum_{1\leq i<j\leq t}n_in_j$ and $n=\sum_{i=1}^{t}n_i$, we have

$$(t-1)n^2-2q(G)t=\sum_{1\leq i<j\leq t}(n_i-n_j)^2.$$

So, (ii) holds.

Taking $z=n_1$, we have

$$z(n-z)+\frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2-q(G)\geq q(G)-q(G)\geq 0.$$

Hence, $n_1\geq\frac{n-\sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i-n_j)^2}}{t}$, which implies that (iii) holds. $\square$

Lemma 5.2.7. *Let $G=K(n_1,n_2,\cdots,n_t)$ and let $H=G-S$ for a set S of s edges of G. If $min\{n_i|i=1,2,\cdots,t\}\geq s+1$, then*

$$s\leq\alpha_{t+1}(H)=\alpha(H,t+1)-\alpha(G,t+1)\leq 2^s-1,$$

$\alpha_{t+1}(H)=s$ if and only if the subgraph induced by any $r\geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H)=2^s-1$ if and only if all s edges in S share a common end-vertex and the other end-vertices belong to the same A_i for some i.

Proof. Obviously, a $(t+1)$-independent partition of $V(G)$ is a $(t+1)$-independent partition of $V(H)$; however, the converse is not always true. So, for a $(t+1)$-independent partition $\mathcal{B}$ of $V(H)$, we have the following two cases.
Case 1. $\mathcal{B}$ is a $(t+1)$-independent partition of $V(G)$.
Case 2. $\mathcal{B}$ is not a $(t+1)$-independent partition of $V(G)$.

Clearly, the number of $(t+1)$-independent partitions $\mathcal{B}$ of $V(H)$ in Case 1 is $\alpha(G, t+1)$. Next we consider the $(t+1)$-independent partitions $\mathcal{B}$ of $V(H)$ in Case 2. Let $\{A_1, A_2, \cdots, A_t\}$ be the unique t-independent partition of $V(G)$, and let $b(H) = \{B_0 | B_0$ is an independent set in H and there are at least two A_i such that $B_0 \cap A_i \neq \phi\}$. Since $min\{n_i | i = 1, 2, \cdots, n_t\} \geq s+1$, we know that $A_i - B_0 \neq \phi$, for any $i = 1, 2, \cdots, t$, where $A_i - B_0$ denotes the subset of A_i obtained by deleting all elements of B_0 from A_i (otherwise, for some i we would have $A_i \subseteq B_0$, and so $|B_0| \geq |A_i| \geq s+1$, which would imply that B_0 is not an independent set in H, since B_0 intersects at least two A_i and we only deleted s edges from G to get H). So, we see that $B_0 \in b(H)$ if and only if $\{B_0, A_1 - B_0, \cdots, A_t - B_0\}$ is a $(t+1)$-independent partition of $V(H)$ of Case 2. Thus, we have $\alpha(H, t+1) = \alpha(G, t+1) + |b(H)|$, i.e., $\alpha(H, t+1) - \alpha(G, t+1) = |b(H)|$. Note that each B_0 of $b(H)$ is composed of pairs of end-vertices of some edges in S. We thus have

$$s \leq |b(H)| = \alpha_{t+1}(H) \leq 2^s - 1.$$

It is not difficult to see that the lower bound s can be reached if and only if the set of all end-vertices of any $r \geq 2$ edges in S is not an independent set and the upper bound $2^s - 1$ can be reached if and only if the set of all end-vertices of any $r \geq 1$ edges is an independent set. □

Lemma 5.2.8. *([71]) Let $n \geq m \geq 2$. Then $K(n,m)$ is χ-unique.*

5.3 The chromaticity of complete tripartite graphs

In this section, we study the chromaticity of complete tripartite graphs by computing the number of triangles of the complete tripartite graphs.

Theorem 5.3.1. *For any integers $n \geq m \geq r \geq 2$, we have $[K(r,m,n)] \subseteq \{K(x,y,z) - S | 1 \leq x \leq y \leq z, m \leq z \leq n, x+y+z = n+m+r, S \subset$*

$E(K(x,y,z))$ and $|S| = xy+xz+yz-nm-nr-mr\}$. In particular, if $z=n$, $K(r,m,n) = K(x,y,z)$.

Proof. Let $G = K(r,m,n)$ and $H \sim G$. We prove that $H \in \{K(x,y,z) - S|1 \leq x \leq y \leq z, m \leq z \leq n, x+y+z = n+m+r, S \subset E(K(x,y,z))$ and $|S| = xy+xz+yz-nm-nr-mr\}$.

From Lemma 5.2.6, we know that there exists a graph $F = K(x,y,z)$ and $S \subset E(F)$ such that $H = F - S$ and $|S| = s$. We may assume that $1 \leq x \leq y \leq z$. Clearly, $s = q(F) - q(G) = xy + xz + yz - nm - nr - mr$ and $x + y + z = n + m + r$. Now we only prove that $m \leq z \leq n$.

Since $H \sim G$, by Lemma 5.2.5 we get $N_A(G) = N_A(H)$. We consider the numbers of triangles in G and H. Without loss of generality, let $S = \{e_1, e_2, \cdots, e_s\} \subset E(F)$. Denote by $N_A(e_i)$ the number of triangles containing the edge e_i in F. It is not hard to see that $N_A(e_i) \leq z$. Then

$$N_A(H) \geq N_A(F) - sz, \tag{5.1}$$

and the equality holds if and only if $N_A(e_i) = z$, for all $e_i \in S$.

Let $\beta = N_A(F) - N_A(G)$. It is obvious that $N_A(F) = xyz$, $N_A(G) = nmr$ and $\beta = xyz - nmr$. So, we have

$$N_A(G) = N_A(F) - \beta. \tag{5.2}$$

Since $N_A(G) = N_A(H)$, from (5.1) and (5.2) it follows that

$$\beta \leq sz. \tag{5.3}$$

Let $f(z) = \beta - sz$. Recalling that $x + y = n + m + r - z$, $\beta = xyz - nmr$ and $s = xy + xz + yz - nm - nr - mr$, we have

$$\begin{aligned} f(z) &= xyz - nmr - [xy + (x+y)z - nm - nr - mr]z \\ &= (z-n)(z-m)(z-r). \end{aligned} \tag{5.4}$$

From the fact that $x + y + z = n + m + r$ and $x \leq y \leq z$, we have $z \geq \frac{n+m+r}{3} \geq r$. Note that if $z = r$, then $n = m = r$, which implies $K(n,m,r) = K(n,n,n)$. Clearly, $z = m = n$. For $z \neq r$, from (5.4) we have that the inequality (5.3) holds if and only if $m \leq z \leq n$. So, $H \in \{K(x,y,z) - S|1 \leq$

$x \le y \le z, m \le z \le n, |S| = xy + xz + yz - nm - nr - mr, x + y + z = n + m + r, S \subset E(K(x, y, z))\}$.

We now prove that $K(x, y, z) = K(n, m, r)$ if $z = n$. Suppose that $z = n$. We distinguish the following cases.

Case 1. $m < y \le n$. Clearly $x + y = m + r$ and $x < r$. Hence $s = xy + xn + yn - nm - nr - mr = xy - mr$. One can show that $s < 0$, for $x < r$ and $y > m$. This contradicts $s \ge 0$.

Case 2. $y = m$. Then $x = r$ and $F = K(r, m, n)$. So, $s = 0$ and $H = G$.

Case 3. $x \le y < m$. Let $\{X_1, X_2, X_3\}$ be the unique 3-independent partition of $K(x, y, n)$ such that $|X_1| = x$, $|X_2| = y$ and $|X_3| = n$. By $f(z) = f(n) = 0$, we have that $\beta = sn$. From (5.1) and (5.2), we have $N_A(G) = N_A(H) = N_A(F) - sn$ and $N_A(e_i) = n$, for all $e_i \in S$. Thus for every edge e_i in S, an end-vertex of e_i belongs to X_1, whereas the other end-vertex of e_i belongs to X_2. Hence $\overline{H}$ contains K_n as its component. Set $\overline{H} = \overline{H_1} \cup K_n$. Then $H = H_1 + O_n$. From Lemma 5.2.1 and $\sigma(H) = \sigma(K(r, m, n))$, we have

$$\sigma(H_1)\sigma(O_n) = \sigma(O_r + O_m)\sigma(O_n).$$

So

$$\sigma(H_1) = \sigma(O_r + O_m),$$

which implies that $P(H_1, \lambda) = P(K_{r,m}, \lambda)$. Hence, from Lemma 5.2.8 and the condition of the theorem, we have $H_1 = K_{r,m}$. So, $y = m$, which contradicts $y < m$. This completes the proof. $\square$

From Theorem 5.3.1, we know that $z = n$ if $n = m$. Therefore a positive answer to Conjecture 5.1.1 is described in the following theorem.

Theorem 5.3.2. *For any integers n and k with $n \ge k + 2 \ge 4$, $K(n-k, n, n)$ is χ-unique.*

Further we can give a new class of χ-unique complete tripartite graphs as follows.

Theorem 5.3.3. *For any integers n and k with $n \ge 2k \ge 4$, $K(n-k, n-1, n)$ is χ-unique.*

Proof. Let $G = K(n-k, n-1, n)$ and $H \sim G$. We prove that $H \cong G$.

By Theorem 5.3.1, we have that $H \in \{K(x,y,z)-S | 1 \le x \le y \le z, n-1 \le z \le n, |S| = xy + xz + yz - 3n^2 + 2nk + 2n - k, x+y+z = 3n-k-1\}$ and $H = G$, for $z = n$. For $z = n-1$, we distinguish the following cases.

Case 1. $y = z = n-1$. Then $H = K(n-k+1, n-1, n-1) - S$. Let $F = K(n-k+1, n-1, n-1)$ and $|S| = s$. Obviously, $s = q(F) - q(G) = k-1$. Let $\alpha_4(H) = \alpha(H,4) - \alpha(F,4)$. From Lemma 5.2.3,

$$\alpha(G,4) = 2^{n-k-1} + 2^{n-2} + 2^{n-1} - 3, \tag{5.5}$$

$$\alpha(H,4) = 2^{n-k} + 2^{n-1} - 3 + \alpha_4(H). \tag{5.6}$$

By the condition, $n-k+1 \ge k+1 \ge s+2$. By Lemma 5.2.7, it follows that $s \le \alpha_4(H) \le 2^s - 1$. Since $k \ge 2$, from (5.5) and (5.6) it follows that

$$\alpha(G,4) - \alpha(H,4) > 2^{n-3} - \alpha_4(H).$$

Remembering that the condition of the theorem and that $s = k-1$, we have immediately that

$$\alpha(G,4) - \alpha(H,4) > 2^{k-1} - 2^{k-1} + 1 \ge 1.$$

This contradicts $\alpha(G,4) = \alpha(H,4)$.

Case 2. $z = n-1$ and $x \le y \le n-2$. By arguments analogous to those used for Case 3 of Theorem 5.3.1, we can obtain that $H = H_1 + O_{n-1}$ and $P(H_1, \lambda) = P(K_{r,n}, \lambda)$. Hence we have $y = n$, which contradicts $y \le n-2$.

From Cases 1 and 2, $z = n$. So, $H = G$. □

5.4 The chromaticity of complete multipartite graphs (I)

As a generalization of Section 5.3, we consider the following problems in this section.

Problem 5.4.1. *Given any integers k and t, where $k \ge 2$, $t \ge 4$ and $n \ge k+2$, is the complete t-partite graph $K(n-k, n, \cdots, n)$ χ-unique?*

Problem 5.4.2. *Given any integers k and t, where $k \geq 2$, $t \geq 4$ and $n \geq k+2$, is the complete t-partite graph $K(n-k, n-1, \cdots, n)$ χ-unique?*

We denote by K_n^+ the graphs with $n+1$ vertices obtained by adding a pendant edge to K_n. By using the inequality on minimum real roots of the adjoint polynomial of a graph, we get an important lemma as follows.

Lemma 5.4.1. *If H is a graph such that $h(H,x) = \prod\limits_{i=1}^{t} h(K_{n_i}, x)$, then H does not contain a $K_{n_t}^+$ as its component, where $n_1 \leq n_2 \leq \cdots \leq n_t$.*

Proof. Suppose that H contains a $K_{n_t}^+$ as its component. From the condition, we have that

$$h(H,x) = \prod_{i=1}^{t} h(K_{n_i}, x). \tag{5.7}$$

From Theorem 3.2.2 and $n_1 \leq n_2 \leq \cdots \leq n_t$, we have that $\beta(H) \leq \beta(K_{n_t}^+) < \beta(K_{n_t}) = \beta\left(\prod\limits_{i=1}^{t} h(K_{n_i}, x)\right)$. This contradicts equation (5.7), and the proof is complete. $\square$

Theorem 5.4.1. *Let $2 \leq n_1 \leq n_2 \cdots \leq n_t$ and $G = K(n_1, n_2, \cdots, n_t)$. If $H \sim G$, then*

(i) $H \in [G] \subset \{K(x_1, x_2, \cdots, x_t) - S | 1 \leq x_1 \leq x_2 \cdots \leq x_t \leq n_t, \sum\limits_{i=1}^{t} x_i = \sum\limits_{i=1}^{t} n_i, S \subset E(K(x_1, x_2, \cdots, x_t))\}$;

(ii) there exists an integer $b \geq 2$ such that $x_1 \leq x_2 \cdots \leq x_b \leq n_b - 1$ and K_{n_i} is a component of $\overline{H}$ for any $i \geq b+1$;

(iii) if $x_i = n_i$, for any $i \geq 3$, then $G = H$.

Proof. (i) Let H be a graph such that $H \sim G$. We only need prove that $H \in \{K(x_1, x_2, \cdots, x_t) - S | 1 \leq x_1 \leq x_2 \cdots \leq x_t \leq n_t, \sum\limits_{i=1}^{t} x_i = \sum\limits_{i=1}^{t} n_i, S \subset E(K(x_1, x_2, \cdots, x_t))\}$.

From Lemma 5.2.6, we know that there exists a complete t–partite graph $F = K(x_1, x_2, \cdots, x_t)$ and $S \subset E(F)$ such that $H = F - S$, $|S| = s = q(F) - q(G)$ and $\sum\limits_{i=1}^{t} x_i = \sum\limits_{i=1}^{t} n_i$. Without loss of generality, we may assume that $1 \leq x_1 \leq x_2 \leq \cdots \leq x_t$. Since $h(\overline{H}) = h(\overline{G})$, from Theorem 2.1.1 and Lemma 5.2.1 we have that

$$h(\overline{H}) = \prod_{i=1}^{t} h(K_{n_i}). \tag{5.8}$$

From (5.8) and Lemma 5.4.1, it is clear that $\overline{H}$ does not contain any $K_{n_t}^{+}$ as its component. So we can see that $x_t \leq n_t$. Hence (i) holds.

(ii) From the proof of (i), it is obvious that $x_t = n_t$ or $x_t \leq n_t - 1$. When $x_t = n_t$, K_{n_t} must be a component of $\overline{H}$ (otherwise, $\overline{H}$ contains a $K_{n_t}^{+}$ as its component). So, we have that if $x_t = n_t$, then $x_1 \leq x_2 \cdots \leq x_{t-1} \leq n_{t-1}$. Let $\overline{H} = \overline{H'} \cup K_{n_t}$. From Theorem 2.1.1 and (5.8), it follows that

$$h(\overline{H'}) = \prod_{i=1}^{t-1} h(K_{n_i}). \tag{5.9}$$

Similarly, we can show that $x_{t-1} < n_{t-1}$ or $x_{t-1} = n_{t-1}$. So, one can see that

(*) if $x_t = n_t$ and $x_{t-1} < n_{t-1}$, then $x_1 \leq x_2 \cdots \leq x_{t-1} \leq n_{t-1} - 1$, while

(**) if $x_t = n_t$ and $x_{t-1} = n_{t-1}$, then $\overline{H'}$ has a component $K_{n_{t-1}}$.

Repeating this procedure for (**), we know that there exists an integer $b \geq 2$ such that $x_1 \leq x_2 \cdots \leq x_b \leq n_b - 1$ and K_{n_i} is a component of $\overline{H}$, for any $i \geq b + 1$.

(iii) From (ii), we see that if $b = 2$, then $\overline{H}$ has $t - 2$ components: K_{n_3}, $K_{n_4}, \cdots, K_{n_t}$. Let $\overline{H} = \overline{H''} \cup \bigcup_{i=3}^{t} K_{n_i}$. Thus, by Theorem 2.1.1 and Lemma 5.2.1, it follows easily that

$$h(\overline{H''}) \prod_{i=3}^{t} h(K_{n_i}) = \prod_{i=1}^{t} h(K_{n_i}). \tag{5.10}$$

From (5.10), we have that

$$h(\overline{H''}) = h(K_{n_1}) h(K_{n_2}). \tag{5.11}$$

By (1.1) and (1.2), one sees that $P(H'', \lambda) = P(K(n_1, n_2), \lambda)$. From Lemma 5.2.8 and the conditions of the theorem, it is not difficult to see that $H'' = K(n_1, n_2)$. So, $\overline{H} = \bigcup_{i=1}^{t} K_{n_i}$, i.e., $G = H$. $\square$

Theorem 5.4.2. *For any positive integers $n \geq k+2$, $k \geq 2$ and $t \geq 3$, the complete t-partite graph $K(n-k, n, n, \cdots, n)$ is χ-unique.*

Proof. Let $G = K(n-k, \overbrace{n, n, \cdots, n}^{t-1})$. Assume that H is a graph such that $H \sim G$. We need prove that $H \cong G$.

By Theorem 5.4.1, we know that there exists a graph $F = K(x_1, x_2, \cdots, x_t)$ and $S \subset E(F)$ such that $H = F - S$ and $|S| = s = q(F) - q(G)$, where $q(F) = \sum\limits_{1 \leq i<j \leq t} x_i x_j$ and $q(G) = (t-1)(n-k)n + \binom{t-1}{2} n^2$, and $1 \leq x_1 \leq x_2 \leq \cdots \leq x_t \leq n$ and $\sum\limits_{i=1}^{t} x_i = nt - k$. Let $S = \{e_1, e_2, \cdots, e_s\} \subset E(F)$, and denote by $N_A(e_i)$ the number of triangles of F containing the edge e_i. It is easy to see that $N_A(e_i) \leq \sum\limits_{i=3}^{t} x_i$, and thus

$$N_A(H) \geq N_A(F) - s \sum_{i=3}^{t} x_i, \tag{5.12}$$

where the equality holds if and only if $N_A(e_i) = \sum\limits_{i=3}^{t} x_i$, for all $e_i \in S$. Let $\alpha = N_A(F) - N_A(G)$. It is obvious that $N_A(F) = \sum\limits_{1 \leq i<j<l \leq t} x_i x_j x_l$ and $N_A(G) = \binom{t-1}{2} n^2 (n-k) + \binom{t-1}{3} n^3$. So, we have that

$$\alpha = \sum_{1 \leq i<j<l \leq t} x_i x_j x_l - \binom{t-1}{2} n^2 (n-k) - \binom{t-1}{3} n^3 \tag{5.13}$$

and

$$N_A(G) = N_A(F) - \alpha. \tag{5.14}$$

From Lemma 5.2.5, $N_A(G) = N_A(H)$, and thus, from (5.12) and (5.14), the following inequality must hold:

$$\alpha \leq s \sum_{i=3}^{t} x_i. \tag{5.15}$$

Assume that $f(x_3, x_4, \cdots, x_t) = \alpha - s \sum\limits_{i=3}^{t} x_i$, and denoted by f_1. Since

$$\alpha = x_1 x_2 \sum_{i=3}^{t} x_i + (x_1 + x_2) \sum_{3 \leq i<j \leq t} x_i x_j + \sum_{3 \leq i<j<l \leq t} x_i x_j x_l$$

$$-\binom{t-1}{2}n^2(n-k)-\binom{t-1}{3}n^3$$

and

$$s=x_1x_2+(x_1+x_2)\sum_{i=3}^{t}x_i+\sum_{3\le i<j\le t}x_ix_j-(t-1)(n-k)n-\binom{t-1}{2}n^2,$$

from $x_1+x_2=nt-k-\sum\limits_{i=3}^{t}x_i$, we have, by calculation, that

$$\begin{aligned}
f_1 &= (x_1+x_2)\left[\sum_{3\le i<j\le t}x_ix_j-\left(\sum_{i=3}^{t}x_i\right)^2\right]+\sum_{3\le i<j<l\le t}x_ix_jx_l\\
&\quad-\sum_{3\le i<j\le t}x_ix_j\sum_{i=3}^{t}x_i+\left[(t-1)(n-k)n+\tbinom{t-1}{2}n^2\right]\sum_{i=3}^{t}x_i\\
&\quad-\tbinom{t-1}{2}n^2(n-k)-\tbinom{t-1}{3}n^3\\
&= (nt-k-\sum_{i=3}^{t}x_i)\left[\sum_{3\le i<j\le t}x_ix_j-\left(\sum_{i=3}^{t}x_i\right)^2\right]+\sum_{3\le i<j<l\le t}x_ix_jx_l\\
&\quad-\sum_{1\le i<j\le t}x_ix_j\sum_{i=3}^{t}x_i+\left[(t-1)(n-k)n+\tbinom{t-1}{2}n^2\right]\sum_{i=3}^{t}x_i\\
&\quad-\tbinom{t-1}{2}n^2(n-k)-\tbinom{t-1}{3}n^3\\
&= \sum_{i=3}^{t}x_i^3+\sum_{3\le i<j\le t}\left(x_i^2x_j+x_ix_j^2\right)+\sum_{3\le i<j<l\le t}x_ix_jx_l-(tn-k)\sum_{i=3}^{t}x_i^2\\
&\quad-(tn-k)\sum_{3\le i<j\le t}x_ix_j+\left[(t-1)(n-k)n+\tbinom{t-1}{2}n^2\right]\sum_{i=3}^{t}x_i\\
&\quad-\tbinom{t-1}{2}n^2(n-k)-\tbinom{t-1}{3}n^3\\
&= \sum_{i=3}^{t}(x_i-n)^2(x_i-n+k)+f_2,
\end{aligned}\tag{5.16}$$

where $f_2=0$, for $t=3$, and

$$\begin{aligned}
f_2 &= \sum_{3\le i<j\le t}\left(x_i^2x_j+x_ix_j^2\right)+\sum_{3\le i<j<l\le t}x_ix_jx_l-(t-3)n\sum_{i=3}^{t}x_i^2\\
&\quad-(tn-k)\sum_{3\le i<j\le t}x_ix_j+\left[(t-4)n^2-(t-3)nk+\tbinom{t-1}{2}n^2\right]\sum_{i=3}^{t}x_i\\
&\quad-\tbinom{t-1}{2}n^2(n-k)-\tbinom{t-1}{3}n^3+(t-2)n^3-(t-2)n^2k,
\end{aligned}\tag{5.17}$$

for $t\ge 4$.

Note that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_t \leq n$. We may assume that $x_i = n - a_i$. Clearly, each a_i is a positive integer and $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_t$. From (5.17), we have, by calculation, that

$$f_2 = k \sum_{3 \leq i < j \leq t} a_i a_j - \sum_{3 \leq i < j \leq t} \left(a_i^2 a_j + a_i a_j^2\right) - \sum_{3 \leq i < j < l \leq t} a_i a_j a_l. \tag{5.18}$$

Since $\sum_{i=1}^{t}(n - a_i) = nt - k$ and $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_t \geq 0$, one can obtain that $k \geq 2a_3 + \sum_{i=3}^{t} a_i$. So, from (5.18) we can get, by simplifying, that

$$f_2 \geq 2a_3 \sum_{3 \leq i < j \leq t} a_i a_j + 2 \sum_{3 \leq i < j < l \leq t} a_i a_j a_l. \tag{5.19}$$

Therefore

$$f_2 \geq 0. \tag{5.20}$$

Recalling that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_t \leq n$, $\sum_{i=1}^{t} x_i = nt - k$ and $k \geq 2$, it is not difficult to see that $x_i > n - k$ for all $i \geq 3$. So,

$$\sum_{i=3}^{n} (x_i - n)^2 (x_i - n + k) \geq 0 \tag{5.21}$$

and the equality holds if and only if $x_i = n$ for all $i \geq 3$. From (5.16), (5.19), (5.20) and (5.21), we have that $f_1 \geq 0$ and the equality holds if and only if $x_i = n$ for all $i \geq 3$. So, (5.15) holds if and only if $x_i = n$ for all $i \geq 3$. By Theorem 5.4.1, it follows that $H = G$. $\square$

With a proof similar to those of Theorems 5.4.2 and 5.3.3, by Theorem 5.4.1 one can show the following theorem.

Theorem 5.4.3. *For any positive integers $n \geq 2k$, $k \geq 2$ and $t \geq 3$, the complete t-partite graph $K(n - k, n - 1, n, \cdots, n)$ is χ-unique.*

5.5 The chromaticity of complete multipartite graphs (II)

In this section, we investigate the chromaticity of complete t-partite graph $K(n_1, n_2, \cdots, n_t)$. Some sufficient conditions for $K(n_1, n_2, \cdots, n_t)$ to be χ-unique are found. Moreover we give a positive answer to Problem 5.1.1.

A class of graphs is said to be *chromatically normal* , if for any two graphs H and G in the class we have that $H \sim G$ implies $H \cong G$.

Theorem 5.5.1. *For a given positive integer t, $\mathcal{K}_t = \{K(n_1, n_2, \cdots, n_t) | n_i$ is a positive integer for $i = 1, 2, \cdots, t\}$ is chromatically normal.*

Proof. Let $H, G \in \mathcal{K}_t$ and $H \sim G$, and let $H = K(m_1, m_2, \cdots, m_t)$ and $G = K(n_1, n_2, \cdots, n_t)$. We prove that $H \cong G$.

Note that $P(H, \lambda) = P(G, \lambda)$ if and only if $\sigma(H, x) = \sigma(G, x)$, i.e., $h(\overline{H}, x) = h(\overline{G}, x)$. From Theorem 2.1.1 and Lemma 5.2.1, we see that

$$\prod_{i=1}^{t} h(K_{m_i}, x) = \prod_{i=1}^{t} h(K_{n_i}, x). \tag{5.22}$$

By (5.22), it is sufficient to show that $\cup_{i=1}^{t} K_{m_i} \cong \cup_{i=1}^{t} K_{n_i}$. We proceed by induction on t. When $t = 1$, the theorem obviously holds.

Suppose $t = k \geq 2$ and the theorem holds when $t \leq k - 1$. Without loss of generality, we assume that $m_1 = max\{m_1, m_2, \cdots, m_t\}$ and $n_1 = max\{n_1, n_2, \cdots, n_t\}$. By Theorem 3.2.2, $\beta(K_n) < \beta(K_{n-1})$, for $n \geq 2$. Thus, we know that the minimum real roots of the left-hand side of equality (5.22) are $\beta(K_{m_1})$, whereas the minimum real roots of the right-hand side of equality (5.22) are $\beta(K_{n_1})$. Hence we have

$$\beta(K_{m_1}) = \beta(K_{n_1}),$$

which implies that $n_1 = m_1$. Eliminating a factor $h(K_{m_1}, x) (= h(K_{n_1}, x))$ from both sides of equality (5.22), we have

$$\prod_{i=2}^{t} h(K_{m_i}, x) = \prod_{i=2}^{t} h(K_{n_i}, x).$$

By the induction hypothesis, we have

$$\cup_{i=2}^{t} K_{m_i} \cong \cup_{i=2}^{t} K_{n_i}.$$

Hence,

$$\cup_{i=1}^{t} K_{m_i} \cong \cup_{i=1}^{t} K_{n_i},$$

as required. □

Theorem 5.5.2. *Let $G = K(n_1, n_2, \cdots, n_t)$ and $n = \sum\limits_{i=1}^{t} n_i$. If $n \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}$, then G is χ-unique.*

Proof. Let H be a graph such that $H \sim G$. Obviously, $\alpha(H, t+1) = \alpha(G, t+1)$. We show that $H \cong G$, by comparing the number of $(t+1)$-independent partitions of H with that of G.

From Lemma 5.2.6, we have that there is a graph $F = K(m_1, m_2, \cdots, m_t)$ such that $\sum\limits_{i=1}^{t} m_i = \sum\limits_{i=1}^{t} n_i = n$ with the property that there is a set S of s edges in F such that $H = F - S$ and $s = q(F) - q(G) \geq 0$. Note that $\alpha_{t+1}(H) = \alpha(H, t+1) - \alpha(F, t+1)$. Clearly, $\alpha_{t+1}(H) \geq 0$. From the condition $n \geq tq(T_{n,t}) - tq(G) + t + \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}$, we have

$$\frac{n - \sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i - n_j)^2}}{t} \geq q(T_{n,t}) - q(G) + 1.$$

So, from Lemmas 5.2.4 and 5.2.6, it follows that $min\{m_i | i = 1, 2, \cdots, t\} \geq q(T_{n,t}) - q(G) + 1 \geq s+1$ and $min\{n_i | i = 1, 2, \cdots, t\} \geq q(T_{n,t}) - q(G) + 1 \geq s+1$. From Lemma 5.2.7, we have $s \leq \alpha_{t+1}(H) \leq 2^s - 1$. Since $\alpha(G, t+1) - \alpha(H, t+1) = \alpha(G, t+1) - \alpha(F, t+1) - \alpha_{t+1}(H)$, from Lemma 5.2.3 we have

$$\alpha(G, t+1) - \alpha(H, t+1) = \sum_{i=1}^{t} 2^{n_i - 1} - \sum_{i=1}^{t} 2^{m_i - 1} - \alpha_{t+1}(H).$$

Without loss of generality, we assume that $min\{n_i | i = 1, 2, \cdots, t\} = n_1$. Then we have

$$\begin{aligned} \alpha(G, t+1) - \alpha(H, t+1) &= 2^{n_1 - 1}\left(\sum_{i=1}^{t} 2^{n_i - n_1} - \sum_{i=1}^{t} 2^{m_i - n_1}\right) - \alpha_{t+1}(H) \\ &= 2^{n_1 - 1} M - \alpha_{t+1}(H), \end{aligned}$$

where $M = \sum_{i=1}^{t} 2^{n_i - n_1} - \sum_{i=1}^{t} 2^{m_i - n_1}$.

We consider the following cases.

Case 1. $M < 0$.

So, $\alpha(G, t+1) - \alpha(H, t+1) < 0$, which contradicts $\alpha(G, t+1) = \alpha(H, t+1)$.

Case 2. $M > 0$.

Subcase 2.1. $min\{m_i | i = 1, 2, \cdots, t\} \geq n_1$.

Then, from the definition of M we see that $M \geq 1$. Remembering that $n_1 \geq q(T_{n,t}) - q(G) + 1 \geq s + 1$ and $s \leq \alpha_{t+1}(H) \leq 2^s - 1$, we have

$$\alpha(G, t+1) - \alpha(H, t+1) = 2^{n_1 - 1} M - \alpha_{t+1}(H) \geq 2^s - (2^s - 1) \geq 1,$$

which also contradicts $\alpha(G, t+1) = \alpha(H, t+1)$.

Subcase 2.2. $min\{m_i | i = 1, 2, \cdots, t\} < n_1$.

Let $\theta = n_1 - min\{m_i | i = 1, 2, \cdots, t\}$. So, $\theta = max\{n_1 - m_i | i = 1, 2, \cdots, t\}$. Then, from the definition of M, it is not difficult to see that $2^\theta M \geq 1$. Since $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i$ and $min\{n_i | i = 1, 2, \cdots, t\} = n_1$ as well as $min\{m_i | i = 1, 2, \cdots, t\} < n_1$, it follows that $max\{m_i | i = 1, 2, \cdots, t\} \geq n_1 + 1$. Hence, $max\{m_i | i = 1, 2, \cdots, t\} - min\{m_i | i = 1, 2, \cdots, t\} \geq \theta + 1$. We have

$$n_1 \geq q(T_{n,t}) - q(G) + 1 = (q(T_{n,t}) - q(F)) + (q(F) - q(G)) + 1.$$

Since $\sum_{i=1}^{t} m_i = n$, from Lemma 5.2.4 we know

$$q(T_{n,t}) - q(F) \geq max\{m_i | i = 1, 2, \cdots, t\} - min\{m_i | i = 1, 2, \cdots, t\} - 1 \geq \theta.$$

Remembering that $q(F) - q(G) = s$, we have $n_1 \geq \theta + s + 1$, i.e., $s \leq n_1 - \theta - 1$. Recalling that $2^\theta M \geq 1$, we obtain

$$\begin{aligned} \alpha(G, t+1) - \alpha(H, t+1) &= 2^{n_1 - \theta - 1} 2^\theta M - \alpha_{t+1}(H) \\ &\geq 2^{n_1 - \theta - 1} - (2^s - 1) \\ &\geq 2^{n_1 - \theta - 1} - (2^{n_1 - \theta - 1} - 1) \\ &\geq 1, \end{aligned}$$

which again contradicts $\alpha(G, t+1) = \alpha(H, t+1)$.

The above contradictions show that we must have $M = 0$. Then, $\alpha(G, t+1) - \alpha(H, t+1) = -\alpha_{t+1}(H)$. Recalling that $\alpha(G, t+1) = \alpha(H, t+1)$, we have

$\alpha_{t+1}(H)=0$. Since $0\le s\le \alpha_{t+1}(H)=0$, we get $s=0$, which implies that $H=K(m_1,m_2,\cdots,m_t)$. Since $H\sim G$, by Theorem 5.5.1 we have $H\cong G$. $\Box$

From Theorem 5.5.2, we can get the following result, which gives an explicit lower bound for the value $min\{n_i|i=1,2,\cdots,t\}$.

Theorem 5.5.3. *Let $G=K(n_1,n_2,\cdots,n_t)$. If $min\{n_i|i=1,2,\cdots,t\}\ge \sum\limits_{1\le i<j\le t}\frac{(n_i-n_j)^2}{2t}+\frac{\sqrt{(t-1)\sum_{1\le i<j\le t}(n_i-n_j)^2}}{t}+1$, then G is χ-unique.*

Proof. Let $n=\sum\limits_{i=1}^{t}n_i=\sum\limits_{i=1}^{t}x_i$. Then, we can show that

$$\sum_{1\le i<j\le t}x_ix_j\le\frac{(t-1)n^2}{2t},$$

where equality holds if and only if t divides n and $x_1=x_2=\cdots=x_t=\frac{n}{t}$. By the definition of $T_{n,t}$ and the above inequality, we know that

$$q(T_{n,t})\le\frac{(t-1)n^2}{2t}.$$

Since

$$\begin{aligned}q(T_{n,t})-q(G) &\le \tfrac{t-1}{2t}n^2-q(G)\\ &= \tfrac{t-1}{2t}(\sum_{i=1}^{t}n_i)^2-\sum_{1\le i<j\le t}n_in_j\\ &= \frac{(t-1)\sum_{i=1}^{t}n_i^2-2\sum_{1\le i<j\le t}n_in_j}{2t}\\ &= \sum_{1\le i<j\le t}\frac{(n_i-n_j)^2}{2t},\end{aligned}$$

from the condition of the theorem, we get

$$n\ge t min\{n_i|i=1,2,\cdots,t\}\ge tq(T_{n,t})-tq(G)+t+\sqrt{(t-1)\sum_{1\le i<j\le t}(n_i-n_j)^2}.$$

From Theorem 5.5.2, we know that G is χ-unique. $\Box$

If one wants to have restrictions on the value $|n_i-n_j|$, one can get the following result, which also answers more than Problem 5.1.1 asked for.

Theorem 5.5.4. *If* $|n_i - n_j| \leq k$ *and* $min\{n_1, n_2, \cdots, n_t\} \geq \frac{tk^2}{4} + \frac{\sqrt{2(t-1)}}{2}k + 1$, *then* $K(n_1, n_2, \cdots, n_t)$ *is* χ*-unique.*

Proof. Assume that $min\{n_1, n_2, \cdots, n_t\} = n'$. Without loss of generality, we may write

$$\{n_1, n_2, \cdots, n_t\} = \{\overbrace{n', \cdots, n'}^{t_0}, \overbrace{n'+1, \cdots, n'+1}^{t_1}, \cdots, \cdots, \overbrace{n'+k, \cdots, n'+k}^{t_k}\}.$$

So, we have

$$\sum_{1 \leq i < j \leq t} \frac{(n_i - n_j)^2}{2t} = \sum_{0 \leq i < j \leq k} \frac{t_i t_j (i-j)^2}{2t}.$$

Since $\sum_{i=0}^{k} t_i = t$, we get

$$\sum_{0 \leq i < j \leq k} \frac{t_i t_j (i-j)^2}{2t} \leq k^2 \sum_{0 \leq i < j \leq k} \frac{t_i t_j}{2t} \leq \left(\begin{array}{c} k+1 \\ 2 \end{array} \right) \frac{k^2 t^2}{2t(k+1)^2} < \frac{tk^2}{4},$$

i.e.,

$$\sum_{1 \leq i < j \leq t} (n_i - n_j)^2 < \frac{t^2 k^2}{2}.$$

From the condition of the theorem and Theorem 5.5.3, the result holds. $\square$

Take $G = K(\overbrace{m, m, \cdots, m}^{a}, \overbrace{m+1, m+1, \cdots, m+1}^{b}, \overbrace{m+2, \cdots, m+2}^{c})$. Let $n = ma + (m+1)b + (m+2)c$ and $t = a + b + c$. It is verified directly that

$$q(T_{n,t}) - q(G) = \min\{a, c\} \leq t/2$$

and

$$\frac{\sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2}}{t} = \frac{\sqrt{(t-1)(ab + 4ac + bc)}}{t} \leq \sqrt{t-1}.$$

Since $t/2 \geq \sqrt{t-1}$ for $t \geq 2$, from Theorem 5.5.2 we have

Theorem 5.5.5. *If* $|n_i - n_j| = 2$ *and* $min\{n_1, n_2, \cdots, n_t\} \geq t + 1$, *then* $K(n_1, n_2, \cdots, n_t)$ *is* χ*-unique, where* $t \geq 2$.

Remarks

In this chapter we focused on the chromaticity of complete multipartite graphs. By comparing the number of triangles of a complete tripartite graph with that of its chromatically equivalent graphs, we got a basic theorem, i.e., Theorem 5.3.1. As byproducts of Theorem 5.3.1, we confirmed a conjecture posed in 1990 by Koh and Teo in Graphs and Combinatorics, see Theorem 5.3.2, and gave a new chromatically unique complete tripartite graph, see Theorem 5.3.3. In Section 5.4, we obtained three results completely similar to those in Section 5.3. Although Theorem 5.4.1 is similar to Theorem 5.3.1, it is very difficult to obtain Theorem 5.4.1, by using the same method as used in Theorem 5.3.1. By using the fact that $\beta(K_n) < \beta(K_{n-1})$, for $n \geq 2$, we proved Theorem 5.4.1. Then, we obtained Theorems 5.4.2 and 5.4.3, by comparing the numbers of triangles of complete t-partite graphs $K(n-k, n, n, \cdots, n)$ and $K(n-k, n-1, n, \cdots, n)$ with those of their chromatically equivalent graphs. In Section 5.5, again by using the fact that $\beta(K_n) < \beta(K_{n-1})$, for $n \geq 2$, we showed that if two complete t-partite graphs are chromatically equivalent, then they are isomorphic, see Theorem 5.5.1. By employing Theorem 5.5.1 and comparing the number of $(t+1)$-independent partitions of a complete t-partite graph with those of its chromatically equivalent graphs, we gave some sufficient conditions for complete multipartite graphs to be chromatically unique, see Theorems 5.5.2, 5.5.3 and 5.5.4 . We solved a problem proposed in 1990 by Koh and Teo in Graphs and Combinatorics, see Theorem 5.5.5. The main results in this chapter are not used in Chapter 6, whereas the basic lemmas in Section 5.2 will play a role in Chapter 6.

Chapter 6

The Chromaticity of Multipartite Graphs

6.1 Introduction

In the preceding chapter, we investigated the chromaticity of complete multipartite graphs and found many new chromatically unique complete multipartite graphs. A natural generalization is to study the chromaticity of general multipartite graphs. Recently, Dong, Koh, Teo, Little and Hendy studied the chromaticity of the bipartite graphs and obtained some remarkable results in [24, 25, 26]. However, there are only few chromatically unique t-partite graphs for $t \geq 3$.

In this chapter, we study the chromaticity of general multipartite graphs. Two basic lemmas are given in Section 6.2. In Sections 6.3 and 6.4, we investigate the chromaticity of the tripartite graphs obtained from a complete bipartite graph by adding some edges between vertices of one of the partition sets in the complete bipartite graph and of the tripartite graphs obtained from a complete tripartite graph by deleting some edges. In the last two sections, we study the chromaticity of 4-partite graphs and of some t-partite graphs, where $t \geq 5$. Many new results are obtained.

Let S be a set of s edges of G. Denote by $G - S$ (simple by $G - s$) the graph obtained from G by deleting all edges in S. Let $K(n, m)$ be a complete bipartite graph with partition sets A_i and A_j such that $|A_i| = n$ and $|A_j| = m$. For $n \geq m \geq s+1$, we denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained by deleting all edges of $K_{1,s}$ from $K(n,m)$ with center in A_i and by $K^{-sK_2}(A_i, A_j)$ the graph obtained by deleting all edges of sK_2 from $K(n,m)$. Let $G = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph with partition sets A_i such that $|A_i| = n_i$, where $i = 1, 2, \cdots, t$. We denote by $\mathcal{G}^{-s}_{n_1,n_2,\cdots,n_t}$ the family $\{G - S | S \subset E(G) \text{ and } |S| = s\}$. By $K(A_i, A_j)$ we denote the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j = 1, 2, \cdots, t$. For $min\{n_1, n_2, \cdots, n_t\} \geq s+1$, we denote by $K^{-K_{1,s}}_{i,j}(n_1, n_2, \cdots, n_t)$ the graph obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting all s edges of $K_{1,s}$ from its subgraph $K(A_i, A_j)$ with center in A_i and others in A_j and by $K^{-sK_2}_{i,j}(n_1, n_2, \cdots, n_t)$ the graph obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting all s edges of sK_2 from its subgraph $K(A_i, A_j)$. Note that $K^{-K_{1,s}}(A_i, A_j) = K^{-K_{1,s}}(A_j, A_i)$ and $K^{-K_{1,s}}_{i,j}(n_1, n_2, \cdots, n_t) = K^{-K_{1,s}}_{j,i}(n_1, n_2, \cdots, n_t)$ if $|A_i| = |A_j|$.

6.2 Some basic results

In this section, we give two important lemmas.

Lemma 6.2.1. *([26]) Let $K(n_1, n_2)$ be a complete bipartite graph with partition sets A_i such that $|A_i| = n_i$, for $i = 1, 2$. If $min\{n_1, n_2\} \geq s+2$, then every $K^{-K_{1,s}}(A_i, A_j)$ is χ-unique, where $i \neq j$ and $i, j = 1, 2$.*

Lemma 6.2.2. *Let $G = K(n_1, n_2, \cdots, n_t)$ with $\sum\limits_{i=1}^{t} n_i = n$ and $n_1 \leq n_2 \leq \cdots \leq n_t$. Suppose that H is a t-partite graph such that $H \sim G - s$, where $S \subset E(G)$ and $|S| = s$. Then there is a graph $F = K(m_1, m_2, \cdots, m_t)$ with $m_1 \leq m_2 \cdots \leq m_t$, and there is a set S' of s' edges in F such that $H = F - S'$, $s' = q(F) - q(G) + s \geq 0$ and F and G satisfy the following conditions:*
(i) $\sum\limits_{i=1}^{t} m_i = \sum\limits_{i=1}^{t} n_i = n$;
(ii) $m_1 \geq \frac{n - \sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2 + 2t(t-1)s}}{t}$;
(iii) $n_1 \geq \frac{n - \sqrt{(t-1)\sum_{1 \leq i < j \leq t}(n_i - n_j)^2 + 2t(t-1)s}}{t}$.

Proof. Since H is a t-partite graph such that $H \sim G - s$, we have $\alpha(H, t) \geq \alpha(G, t) = 1$ and $\alpha(H, r) = 0$, for $0 \leq r \leq t - 1$. Then $V(H)$ has at least one t-independent partition, say $\{A_1, A_2, \cdots, A_t\}$, and H is a t-partite graph. Let $|A_i| = m_i$, $i = 1, 2, \cdots, t$. Then there is a set S' of s' edges in $F = K(m_1, m_2, \cdots, m_t)$ such that $H = K(m_1, m_2, \cdots, m_t) - S' = F - S'$. Remembering that $p(H) = p(G - s)$ and $q(H) = q(G - s)$, it follows that $\sum_{i=1}^{t} m_i = \sum_{i=1}^{t} n_i = n$ and $s' = q(F) - q(G) + s \geq 0$, which implies that (i) is true.

(ii) Let z denote the minimum value of m_1 such that $s' \geq 0$. Then $q(K(z, m_2, m_3, \cdots, m_t)) - q(G) + s \geq 0$ for some $(m_2, m_3, \cdots, m_t)$. Denote by $K(z, y_2, \cdots, y_t)$ the complete t-partite graphs with $z \leq y_2 \leq y_3 \cdots \leq y_t$ and $|y_i - y_j| \leq 1$, for $i, j = 2, 3, \cdots, t$, where $\sum_{i=2}^{t} y_i = n - z$. With an argument similar to that of Lemma 5.2.7, we have that $q(K(z, y_2, \cdots, y_t)) \geq q(K(z, m_2, m_3, \cdots, m_t))$, for all $(m_2, m_3, \cdots, m_t)$, and that

$$q(K(z, y_2, \cdots, y_t)) \leq z(n - z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2.$$

Since $s' \geq 0$, we have that z satisfies the following inequality

$$z(n - z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2 - q(G) + s \geq 0.$$

By solving the above inequality, we have

$$\frac{n - \sqrt{(t-1)((t-1)n^2 - 2q(G)t + 2st)}}{t} \leq z$$

$$\leq \frac{n + \sqrt{(t-1)((t-1)n^2 - 2q(G)t + 2st)}}{t}.$$

Since $q(G) = \sum_{1 \leq i < j \leq t} n_i n_j$ and $n = \sum_{i=1}^{t} n_i$, we have

$$(t-1)n^2 - 2q(G)t = \sum_{1 \leq i < j \leq t} (n_i - n_j)^2.$$

So, (ii) holds.

Taking $z = n_1$, we have

$$z(n-z) + \frac{(t-1)(t-2)}{2}\left(\frac{n-z}{t-1}\right)^2 - q(G) + s \geq s \geq 0.$$

Hence, $n_1 \geq \frac{n-\sqrt{(t-1)\sum_{1\leq i<j\leq t}(n_i-n_j)^2+2t(t-1)s}}{t}$, which implies that (iii) holds.

$\square$

6.3 The chromaticity of tripartite graphs (I)

For $n \geq m \geq 2$, let $K(n,m)$ be a complete bipartite graph with partition sets A_1 and A_2 such that $|A_1| = n$ and $|A_2| = m$. We denote by $K(n,m)+s$ the graph obtained by adding s edges between vertices of one of the partition sets A_1 and A_2. Denote $K(n,n)+s$ simply by $K^{+s}(n,n)$. In 1990, Teo and Koh [71] have shown that $K(n,m)-1$ is χ-unique. They also proposed the following problem.

Problem 6.3.1. *([71]) For $n \geq m \geq 2$, study the chromaticity of $K(n,m)+1$.*

We investigate a more general problem.

Problem 6.3.2. *For $n \geq m \geq 2$ and $s \geq 1$, study the chromaticity of $K(n,m)+s$.*

In this section, we obtain all chromatically equivalent classes of $K^{+s}(n,n)$, when $n \geq s+2$. A sufficient and necessary condition for $K^{+s}(n,n)$ to be χ-unique is obtained, where $n \geq s+2$. Moreover, we give a partial answer to the above problems.

Theorem 6.3.1. *For $n \geq s+2$ and $s \geq 1$, let S be a set of s edges in K_n and let $< S >$ be a bipartite graph. Then $[K^{+s}(n,n)] = \{O_n + G | G \in [\overline{K_n - s}]\}$ and $K^{+s}(n,n)$ is χ-unique if and only if $\overline{K_n - s}$ is χ-unique, where $O_n + G$ is the join graph of O_n and G.*

Proof. Let $Y = K^{+s}(n,n)$. By the condition of the theorem, there is a set S of s edges in K_n such that $Y = K(n,n)+s$ and $< S >$ is a bipartite graph. Clearly, Y is a tripartite graph. Let $Y' = \overline{K_n - s}$. Then $Y \cong O_n + Y'$. Suppose that $H \sim Y$. It suffices to prove that $H \cong O_n + H'$ and $H' \sim Y'$.

By Lemma 6.2.2, there is a graph $F = K(m_1, m_2, m_3)$ such that $H = F - s'$ and $s' = |E(F)| - |E(Y)| = m_1m_2 + m_1m_3 + m_2m_3 - n^2 - s$, where $m_1 + m_2 + m_3 = 2n$. Without loss of generality, let $m_1 \le m_2 \le m_3$ and $S' = \{e_1, e_2, \cdots, e_{s'}\}$. We denote by $N_A(e_i)$ the number of triangles containing the edge e_i in F. Then $N_A(e_i) \le m_3$. So, we have

$$N_A(H) \ge N_A(F) - s'm_3 = m_1m_2m_3 - s'm_3 \tag{6.1}$$

and equality holds if and only if $N_A(e_i) = m_3$ for each $e_i \in S'$.

By Lemma 5.2.5, $N_A(Y) = N_A(H)$. Since $N_A(Y) = ns$, by (6.1) the following inequality must be true.

$$m_1m_2m_3 - s'm_3 \le ns. \tag{6.2}$$

Let $f(m_3) = m_1m_2m_3 - s'm_3 - ns$. Recalling that $s' = m_1m_2 + m_1m_3 + m_2m_3 - n^2 - s$ and $m_1 + m_2 = 2n - m_3$, we obtain by calculation that

$$f(m_3) = m_3(m_3 - n)^2 + (m_3 - n)s. \tag{6.3}$$

By (6.3), $f(m_3) > 0$ when $m_3 > n$. When $m_3 \le n$, we have, from (6.2), that

$$f(m_3) = (m_3 - n)(m_3^2 - m_3n + s). \tag{6.4}$$

Since $m_1 \le m_2 \le m_3$ and $m_1+m_2+m_3 = 2n$, we have that $m_3 \ge 2n/3$ and $2n/3 \le m_3 \le n$. It is not difficult to see that $m_3^2 - m_3n + s < 0$ when $n \ge s+2$ and $m_3 = 2n/3$, or $m_3 = n-1$. Hence, for $n \ge s+2$ and $2n/3 \le m_3 \le n-1$, we have $m_3^2 - m_3n + s < 0$. In fact, when $n \ge s+2$ and $2n/3 \le m_3 \le n$, $f(m_3) \ge 0$ and the equality holds if and only if $m_3 = n$. So, (6.2) holds if and only if $m_3 = n$, and $m_1 \le m_2 < m_3 = n$. By (6.1), $N_A(e_i) = n$ for $e_i \in S'$. Suppose that $V(F)$ has a unique 3-independent partition $\{V_1, V_2, V_3\}$ with $|V_i| = m_i$, $i = 1, 2, 3$. Then for each edge $e_i \in S'$, one edge-vertex of e_i belongs to V_1 and another edge-vertex of e_i belongs to V_2 (otherwise $N_A(e_i) < n$). Therefore $\overline{H}$ has a component K_n, so, $H = O_n + H'$. By Lemma 5.2.1,

$$\sigma(Y) = \sigma(O_n)\sigma(Y'), \quad \sigma(H) = \sigma(O_n)\sigma(H').$$

Since $\sigma(Y) = \sigma(H)$, we have that $\sigma(Y') = \sigma(H')$, i.e., $Y' \sim H'$. So, $H \cong O_n + H'$ and $H' \in [\overline{K_n - s}]$. Obviously, $H \cong Y$ if and only if $\overline{K_n - s}$ is χ-unique. This completes the proof of the theorem. $\square$

Lemma 6.3.1. *([66, 77]) Let G be a connected graph and $H = G \cup rK_1$. If $r \geq 1$, then H is χ-unique if and only if G is a χ-unique graph without cut-vertex.*

From Theorem 6.3.1 and Lemma 6.3.1, we have

Theorem 6.3.2. *For $n \geq s+2$ and $s \geq 1$, let S be a set of s edges in K_n and let $< S >$ be a bipartite graph. Then $K^{+s}(n,n)$ is χ-unique if and only if $< S >$ is a χ-unique graph without cut-vertex.*

From [43, 71], we can find the following chromatically unique bipartite graphs:
(i) [43] For $n \geq 4$ and n is even, C_n is χ-unique.
(ii) [71] For $n \geq m \geq 2$, $K(n,m)$ is χ-unique.

By $K^{+G}(r,r)$ we denote the graph obtained by adding all edges of G between vertices of one of the partition sets in $K(r,r)$. From Lemma 6.2.1 and Theorem 6.3.1, we have

Corollary 6.3.1. *Let $G \in \{K(n,m)|n \geq m \geq 2\} \cup \{C_n|n \geq 4 \text{ and } n \text{ is even}\}$. If $r \geq |E(G)|+2$, then $K^{+G}(r,r)$ is χ-unique.*

Corollary 6.3.2. *For $n \geq 3$, $K(n,n)+1$ is χ-unique.*

6.4 The chromaticity of tripartite graphs (II)

In this section we study the chromaticity of tripartite graphs obtained from a complete tripartite graph by deleting some edges.

Let i, k, m be positive integers and $0 \leq i \leq k$. Set

$$f(\mathbf{x}) = f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} + 2^{x_3} - 2^m - 2^{m+i} - 2^{m+k}|, \tag{6.5}$$

where $x_1 + x_2 + x_3 = 3m + i + k$ and x_h is a positive integer for $h = 1, 2, 3$.

Theorem 6.4.1. *Let $f(\mathbf{x})$ be the function defined by (6.5). Then $f(\mathbf{x}) \geq 2^m$ except for the following cases:*
(i) $f(\mathbf{x}) = 0$ if $\mathbf{x} = (m, m+i, m+k)$;
(ii) $f(\mathbf{x}) = 2^{m-1}$ if $\mathbf{x} = (m-1, m+1, m+k)$ and $i = 0$, $k \geq 0$.

Proof. Without loss of generality, let $x_1 \le x_2 \le x_3$. We distinguish the following cases:

Case 1. $x_3 \ge m+k+2$. Since $2^{x_3} \ge 2^{m+k+2} = 4 \times 2^{m+k}$, by (6.5)

$$f(x_1, x_2, x_3) > 2^{m+k} \ge 2^m.$$

Case 2. $x_3 = m+k+1$. Then $x_1 + x_2 = 2m+i-1$. For $2^{x_3} \ge 2^{m+k+1} = 2 \times 2^{m+k}$, by (6.5),

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} + 2^{m+k} - 2^m - 2^{m+i}|. \tag{6.6}$$

Case 2.1. $i = k$. So, $x_1 + x_2 = 2m+i-1$. By (6.6),

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} - 2^m|. \tag{6.7}$$

For $i \ge 0$, we have that $x_2 \ge m+1$, or $x_1 = x_2 = m$, or $x_1 = m-1$ and $x_2 = m$. So, if $x_2 \ge m+1$, or $x_1 = x_2 = m$, by (6.7), $f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} - 2^m| \ge 2^m$. If $x_1 = m-1$ and $x_2 = m$, from (6.7) we have $f(x_1, x_2, x_3) = 2^{m-1}$ and $i = k = 0$, which is a special case of (ii).

Case 2.2. $i \le k-1$. From $2^{m+k} \ge 2 \times 2^{m+i}$ and $x_1 + x_2 = 2m+i-1$, we have $x_2 \ge m$. By (6.6), $f(x_1, x_2, x_3) \ge 2^m$.

Case 3. $x_3 = m+k$. Then $x_1 + x_2 = 2m+i$. By (6.5),

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i}|. \tag{6.8}$$

Case 3.1. $x_2 \ge m+i+2$. So, $2^{x_2} = 4 \times 2^{m+i}$. By (6.8), $f(x_1, x_2, x_3) > 2^m$.

Case 3.2. $x_2 = m+i+1$. Then $x_1 = m-1$. By (6.8),

$$f(x_1, x_2, x_3) = |2^{m-1} + 2^{m+i} - 2^m|.$$

For $i \ge 1$, $f(x_1, x_2, x_3) = |2^{m-1} + 2^{m+i} - 2^m| \ge 2^m$; for $i = 0$, $f(x_1, x_2, x_3) = |2^{m-1} + 2^{m+i} - 2^m| = 2^{m-1}$, that is, $x_1 = m-1, x_2 = m+1$ and $x_3 = m+k$, which is (ii).

Case 3.3. $x_2 = m+i$. Then $x_1 = m$. So, $f(x_1, x_2, x_3) = 0$.

Case 3.4. $x_2 = m+i-1$. Clearly $x_1 = m+1$. Since $x_1 \le x_2$, we have $i \ge 2$. By (6.8),

$$f(x_1, x_2, x_3) = |2^{m+1} + 2^{m+i-1} - 2^m - 2^{m+i}| = |2^m - 2^{m+i-1}| \ge 2^m.$$

Case 3.5. $x_2 \leq m+i-2$. By $x_1 \leq x_2$ and (6.8),

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} - 2^m - 4 \times 2^{m+i-2}| > 2^m.$$

Case 4. $x_3 = m+k-1$. So, $x_1 + x_2 = 2m+i+1$. By (6.5),

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i} - 2^{m+k-1}|. \tag{6.9}$$

Case 4.1. $x_2 = x_3 = m+k-1$. So, $x_1 = m+i+2-k$. By (6.9),

$$f(x_1, x_2, x_3) = |2^{m+i+2-k} - 2^m - 2^{m+i}|.$$

For $k \geq 2$, $f(x_1, x_2, x_3) \geq 2^m$; For $k=1$ and $i=1$, $f(x_1, x_2, x_3) \geq 2^m$; For $k=1$ and $i=0$, or $k=i=0$, $x_1 > x_2$, which contradicts $x_1 \leq x_2$.

Case 4.2. $x_2 \leq x_3 - 1$.

Clearly, $2^{m+k-1} = 2^{x_3} \geq 2 \times 2^{x_2}$. By (6.9), $f(x_1, x_2, x_3) \geq 2^m$.

Case 5. $x_3 \leq m+k-2$. As $2^{m+k+2} = 4 \times 2^{x_3}$, we have $f(x_1, x_2, x_3) > 2^m$.

This completes the proof of the theorem. $\square$

The proofs of the following three theorems are typical for the proofs of further theorems on χ-uniqueness in coming sections. For that reason, the proofs are given in a more elaborate way.

Theorem 6.4.2. *Let $n_1 \leq n_2 \leq n_3$ with $n = n_1 + n_2 + n_3$ and $s \geq 1$. If $n > \frac{1}{2}\sum_{1\leq i<j\leq 3}(n_i - n_j)^2 + \sqrt{2\sum_{1\leq i<j\leq 3}(n_i - n_j)^2 + 12s} + 3s + 3$, then $\mathcal{G}^{-s}_{n_1,n_2,n_3}$ is χ-closed.*

Proof. Suppose that $Y \in \mathcal{G}^{-s}_{n_1,n_2,n_3}$ and $Y = G - s$, where $G = K(n_1, n_2, n_3)$. Let H be a graph with $H \sim Y$. It suffices to prove that $H \in \mathcal{G}^{-s}_{n_1,n_2,n_3}$.

Note that Y is a tripartite graph. By Lemma 6.2.2, there is a graph $F = K(m_1, m_2, m_3)$ such that $H = F - s'$ and $s' = q(F) - q(G) + s$, where $m_1 \leq m_2 \leq m_3$. Set

$$f(x_1, x_2, x_3) = |2^{x_1} + 2^{x_2} + 2^{x_3} - 2^{n_1} - 2^{n_2} - 2^{n_3}|.$$

Note that

$$\alpha(G - s, 4) = \sum_{i=1}^{3} 2^{n_i - 1} - 3 + \alpha_4(G - s)$$

and

$$\alpha(F-s',4)=\sum_{i=1}^{3}2^{m_i-1}-3+\alpha_4(F-s').$$

By Lemma 5.2.5, $\alpha(H,4)=\alpha(G-s,4)=\alpha(F-s',4)$. So,

$$\alpha(F-s',4)-\alpha(G-s,4)=\sum_{i=1}^{3}2^{m_i-1}-\sum_{i=1}^{3}2^{n_i-1}+\alpha_4(F-s')-\alpha_4(G-s)$$

$$=\begin{cases}\frac{1}{2}f(m_1,m_2,m_3)+\alpha_4(F-s')-\alpha_4(G-s), & \text{if } \alpha(F,4)\geq\alpha(G,4),\\ -\frac{1}{2}f(m_1,m_2,m_3)+\alpha_4(F-s')-\alpha_4(G-s), & \text{if } \alpha(F,4)<\alpha(G,4).\end{cases}$$

By Theorem 6.4.1,

$$f(m_1,m_2,m_3)\begin{cases}=0, & \text{if } (m_1,m_2,m_3)=(n_1,n_2,n_3),\\ =2^{n_1-1}, & \text{if } (m_1,m_2,m_3)=(n_1-1,n_1+1,n_3) \text{ and } n_1=n_2,\\ \geq 2^{n_1}, & \text{otherwise.}\end{cases}$$

Since $\frac{1}{6}\sum\limits_{1\leq i<j\leq 3}(n_i-n_j)^2\geq q(T_{n,3})-q(G)$, by the condition it follows that

$$\frac{n-\sqrt{2\sum\limits_{1\leq i<j\leq t}(n_i-n_j)^2+12s}}{3}>q(T_{n,3})-q(G)+s+1.$$

From Lemmas 5.2.4 and 6.2.2, $m_1>q(T_{n,3})-q(G)+s+1\geq s'+2$ and $n_1>q(T_{n,3})-q(G)+s+1\geq s+2$. Thus, by Lemma 5.2.7, we have

$$0\leq s'\leq\alpha_4(F-s')\leq 2^{s'}-1$$

and

$$0\leq s\leq\alpha_4(G-s)\leq 2^{s}-1.$$

As in Theorem 6.4.1, we prove that $H\in\mathcal{G}^{-s}_{n_1,n_2,n_3}$ by distinguishing the following cases:

Case 1. $(m_1,m_2,m_3)=(n_1-1,n_2+1,n_3)$ and $n_1=n_2$.

Obviously, $\alpha(F,4)-\alpha(H,4)=2^{n_1-2}$. So, we have

$$\alpha(F-s',4)-\alpha(G-s,4)=2^{n_1-2}+\alpha_4(F-s')-\alpha_4(G-s).$$

Since $n_1 \geq s+2$, $0 \leq s' \leq \alpha_4(F-s') \leq 2^{s'}-1$ and $0 \leq s \leq \alpha_4(G-s) \leq 2^s-1$, we have

$$\alpha(F-s',4) - \alpha(G-s,4) \geq 2^{n_1-2} + \alpha_4(F-s') - 2^{n_1-2} + 1 \geq 1,$$

which contradicts $\alpha(F-s',4) = \alpha(G-s,4)$.

Case 2. $(m_1, m_2, m_3) \neq (n_1-1, n_1+1, n_3)$ and $n_1 = n_2$, or $(m_1, m_2, m_3) \neq (n_1, n_2, n_3)$.

Note that $n_1 \geq q(T_{n,3}) - q(G) + s + 2 \geq [(T_{n,3}) - q(F)] + [q(F) - q(G) + s] + 2 \geq s' + 2$. Since $n_1 \geq s+2$, from Theorem 6.4.1 we have

$$\begin{array}{rcl} \alpha(F-s',4) - \alpha(G-s,4) & \geq & 2^{n_1-1} + \alpha_4(F-s') - \alpha_4(G-s) \\ & > & 2^s + \alpha_4(F-s') - 2^s + 1 \geq 1, \end{array}$$

or

$$\begin{array}{rcl} \alpha(F-s',4) - \alpha(G-s,4) & \leq & -2^{n_1-1} + \alpha_4(F-s') - \alpha_4(G-s) \\ & < & -2^{s'} + 2^{s'} - 1 - \alpha_4(G-s) \leq -1. \end{array}$$

This contradicts $\alpha(F-s',4) = \alpha(G-s,4)$.

From Cases 1 and 2, $(m_1, m_2, m_3) = (n_1, n_2, n_3)$. Furthermore, $F = G$ and $s = s'$. So, $H \in \mathcal{G}^{-s}_{n_1,n_2,n_3}$. □

Corollary 6.4.1. *If $j \leq k$ and $m > \frac{k^2}{3} + \frac{2}{3}\sqrt{k^2+3s} + s + 1$, then $\mathcal{G}^{-s}_{m,m+j,n+k}$ is χ-closed, where $s \geq 1$.*

Proof. Let $n_1 = m, n_2 = m+j$ and $n_3 = m+k$. For $\sum\limits_{1 \leq i < j \leq 3} (n_i - n_j)^2 = j^2 + k^2 + (k-j)^2 \leq 2k^2$, we have that $n \geq 3m > k^2 + 2\sqrt{k^2+3s} + 3s + 3 > \frac{1}{2} \sum\limits_{1 \leq i < j \leq 3} (n_i - n_j)^2 + \sqrt{2 \sum\limits_{1 \leq i < j \leq t} (n_i - n_j)^2 + 12s} + 3s + 3$. By Theorem 6.4.2, the result follows. □

Theorem 6.4.2 and Corollary 6.4.1 gave some sufficient conditions for a family $\mathcal{G}^{-s}_{n_1,n_2,n_3}$ to be χ-closed. Let $min\{n_1, n_2, n_3\} \geq s+1$ and $H \in \mathcal{G}^{-s}_{n_1,n_2,n_3}$. We consider the number of 4-independent partitions of H. Note that $\alpha(H,4) = \alpha(G,4) + \alpha_4(H)$, where $\alpha(G,4) = 2^{n_1-1} + 2^{n_2-1} + 2^{n_3-1} - 3$ and $s \leq \alpha_4(H) \leq 2^s - 1$. By Lemma 5.2.7, one sees that

(i) $\alpha_4(H) = 2^s - 1$ if and only if all s edges in S share a common end-vertex and the other end-vertices belong to the same A_i for some i, i.e., H is one of the following graphs:

$$K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3),\ \text{for } i \neq j, i, j = 1, 2, 3.$$

(ii) $\alpha_4(H) = s$ if and only if the subgraph induced by any r $(r \geq 2)$ edges in S is not a complete multipartite graph. There are many graphs H such that $\alpha_4(H) = s$.

We investigate the chromatic uniqueness of all graphs H with $\alpha_4(H) = 2^s - 1$ and of one of the graphs H with $\alpha_4(H) = s$ in the following theorems.

Theorem 6.4.3. *Let $n_1+n_2+n_3 = n$ and $s \geq 1$. If $n > \frac{1}{2}\sum_{1\leq i<j\leq 3}(n_i - n_j)^2 + \sqrt{2\sum_{1\leq i<j\leq 3}(n_i - n_j)^2 + 12s} + 3s + 3$, then $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3)$ is χ-unique, where $i \neq j$ and $i, j = 1, 2, 3$.*

Proof. Let $G = K(n_1, n_2, n_3)$. Suppose that $F \in \{K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3) | i \neq j$ and $i, j = 1, 2, 3\}$ and $H \sim F$. Now we prove that $H = F$.

By the condition of the theorem and Theorem 6.4.2,

$$H \in \mathcal{G}^{-s}_{n_1,n_2,n_3}.$$

Note that $\alpha(H, 4) = \alpha(G, 4) + \alpha_4(H)$, where $\alpha(G, 4) = 2^{n_1-1} + 2^{n_2-1} + 2^{n_3-1} - 3$. Since $\alpha_4(H) = \alpha_4(F) = 2^s - 1$, it follows, from Lemma 5.2.7, that

$$H \in \{K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3) | i \neq j,\ i, j = 1, 2, 3\}.$$

By Lemma 5.2.5, $N_A(H) = N_A(F)$. Therefore we consider the numbers of triangles of H and F. According to the numbers of triangle of F, we partition $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3)$ into three classes:

Type 1: $F \in \mathcal{G}_1 = \{K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3), K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3\},\ N_A(F) = n_1n_2n_3 - sn_1$;

Type 2: $F \in \mathcal{G}_2 = \{K_{1,3}^{-K_{1,s}}(n_1, n_2, n_3), K_{3,1}^{-K_{1,s}}(n_1, n_2, n_3)\},\ N_A(F) = n_1n_2n_3 - sn_2$;

Type 3: $F \in \mathcal{G}_3 = \{K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3), K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3)\},\ N_A(F) = n_1n_2n_3 - sn_3$.

It is not hard to see that $F, H \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. If $F \in \mathcal{G}_a$, $H \in \mathcal{G}_b$ and $a \neq b$, by $N_A(H) = N_A(F)$, we have $n_a = n_b$. Note that if $n_i = n_j$, then, for any $l \neq i, j$, we have

$$K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3),$$

and

$$K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3).$$

Hence $n_a = n_b$ implies that $\mathcal{G}_a = \mathcal{G}_b$ and F and H belong to the same set $\mathcal{G}_i$, where $i = 1, 2, 3$.

Without loss of generality, let $F \in \mathcal{G}_1$ and $H \in \mathcal{G}_1$. For $n_2 = n_3$, $K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3) = K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3)$, we need only prove that if $n_2 \neq n_3$, then

$$P(K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3), \lambda) \neq P(K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3), \lambda).$$

Since $P(H_1, \lambda) = P(H_2, \lambda)$ if and only if $\sigma(H_1, x) = \sigma(H_2, x)$, for two graphs H_1 and H_2, we consider the σ-polynomials of F and H. By Lemma 5.2.1,

$$\sigma(K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3), x) = \sigma(K^{-K_{1,s}}(A_2, A_3), x)\sigma(K_{n_1}, x),$$

$$\sigma(K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3), x) = \sigma(K^{-K_{1,s}}(A_3, A_2), x)\sigma(K_{n_1}, x).$$

By the proof of Theorem 6.4.2, one sees that $min\{n_1, n_2, n_3\} \geq s + 2$. By Lemma 6.2.1, if $n_2 \neq n_3$, we have

$$\sigma(K^{-K_{1,s}}(A_2, A_3), x) \neq \sigma(K^{-K_{1,s}}(A_3, A_2), x).$$

So,

$$\sigma(K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3), x) \neq \sigma(K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3), x),$$

i.e.,

$$P(K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3), \lambda) \neq P(K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3), \lambda).$$

Hence $F = H$ and both $K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3)$ and $K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3)$ are χ-unique.

Similarly, one can prove cases in which $F \in \mathcal{G}_i$ and $H \in \mathcal{G}_i$, $i = 2, 3$. $\square$

Similar to the derivation of Corollary 6.4.1, by Theorem 6.4.3, we have

Corollary 6.4.2. *If $n_1 \le n_2 \le n_3$, $n_3 - n_1 = k$ and $n_1 > \frac{k^2}{3} + \frac{2}{3}\sqrt{k^2+3s} + s + 1$, then $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3)$ is χ-unique, where $s \ge 1$, $i \ne j$, $i, j = 1, 2, 3$.*

Let $G = K(n_1, n_2, n_3)$ with $n_1 \le n_2 \le n_3$ and let A_1, A_2 and A_3 be three partition sets with $|A_i| = n_i$, where $i = 1, 2, 3$. We denote by $H_{n_1,n_2,n_3}^{-sK_2}$ the graph obtained by deleting all edges of sK_2 from $K(A_1, A_2)$ in G.

Theorem 6.4.4. *Suppose $n_1 + n_2 + n_3 = n$ and $s \ge 1$. If $n_1 \le n_2 < n_3$ and $n > \frac{1}{2}\sum_{1\le i<j\le 3}(n_i - n_j)^2 + \sqrt{2\sum_{1\le i<j\le 3}(n_i - n_j)^2 + 12s} + 3s + 3$, then $H_{n_1,n_2,n_3}^{-sK_2}$ is χ-unique.*

Proof. Suppose $F \sim H_{n_1,n_2,n_3}^{-sK_2}$. It is sufficient to prove that $F = H_{n_1,n_2,n_3}^{-sK_2}$.

By Theorem 6.4.2, $F \in \mathcal{G}_{n_1,n_2,n_3}^{-s}$. Let $S = \{e_1, e_2, e_3, \cdots, e_s\}$ and let $N_A(e_i)$ denote the number of triangles containing e_i in $K(n_1, n_2, n_3)$. Then $N_A(e_i) \le n_3$. By $n_1 \le n_2 < n_3$, $N_A(e_i) = n_3$ if and only if e_i is an edge of subgraph $K(A_1, A_2)$. So,

$$N_A(F) \ge n_1n_2n_3 - \sum_{i=1}^{s} N_A(e_i) \ge n_1n_2n_3 - sn_3$$

and equality holds if and only if all edges of S are edges of subgraph $K(A_1, A_2)$. Note that $N_A(F) = n_1n_2n_3 - sn_3$ and $\alpha_4(F) = \alpha_4(H_{n_1,n_2,n_3}^{-sK_2}) = s$. From Lemma 5.2.7, we have $F = H_{n_1,n_2,n_3}^{-sK_2}$. □

By Theorem 6.4.4, the following is obtained easily.

Corollary 6.4.3. *If $n_1 \le n_2 < n_3$, $n_3 - n_1 = k$ and $n_1 > \frac{k^2}{3} + \frac{2}{3}\sqrt{k^2+3s} + s + 1$, then $H_{n_1,n_2,n_3}^{-sK_2}$ is χ-unique, where $s \ge 1$.*

6.5 The chromaticity of 4-partite graphs

With the same method as used in the preceding section, in this section we investigate the chromaticity of 4-partite graphs obtained from a complete 4-partite graph by deleting some edges. First we give the lower bounds for the function $g(\mathbf{x})$, which is defined by

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} + 2^{x_4} - 2^m - 2^{m+i} - 2^{m+j} - 2^{m+k}|, \tag{6.10}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, x_h is a positive integer for $h = 1, 2, 3, 4$ and $x_1 + x_2 + x_3 + x_4 = 4m + i + j + k$.

Theorem 6.5.1. *Let $g(\mathbf{x})$ be the function defined by (6.10). Then $g(\mathbf{x}) \geq 2^m$ except for the following cases:*
(i) $g(\mathbf{x}) = 0$ if $\mathbf{x} = (m, m+i, m+j, m+k)$;
(ii) $g(\mathbf{x}) = 2^{m-k+1}$ if $\mathbf{x} = (m-k+1, m+2, m+k-1, m+k-1)$ and $i = 0$, $j = 1$ and $k \geq 3$;
(iii) $g(\mathbf{x}) = 2^{m-1}$ if $\mathbf{x} = (m-1, m+1, m+j, m+k)$ and $i = 0$, or $\mathbf{x} = (m-1, m+k-1, m+k-1, m+k-1)$ and $i = j = k-2$ and $k \geq 2$;
(iv) $2^{m-1} < g(\mathbf{x}) < 2^m$ if $\mathbf{x} = (m-b, m+i+1, m+b+i, m+b+i)$, and $i = j$ and $k = b+i+1$, where b is a positive integer and $b \geq 2$.

Proof. Without loss of generality, assume that $x_1 \leq x_2 \leq x_3 \leq x_4$. We distinguish the following cases.

Case 1. $x_4 \geq m+k+3$. Since $2^{x_4} \geq 2^{m+k+3} = 8 \times 2^{m+k}$, it is clear from (6.10) that $g(\mathbf{x}) \geq 2^{m+k} > 2^m$.

Case 2. $x_4 = m+k+2$. So, $x_1 + x_2 + x_3 = 3m+i+j-2$.

As $x_1+x_2+x_3 = 3m+i+j-2$, $x_3 \geq m$. Clearly, $2^{x_4}+2^{x_3} \geq 4\times 2^{m+k}+2^m$. Hence, by (6.10), we have $g(\mathbf{x}) \geq 2^{m+k} > 2^m$.

Case 3. $x_4 = m+k+1$. Then, $x_1 + x_2 + x_3 = 3m+i+j-1$.

Subcase 3.1. $j = k$. From (6.10), it follows that

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} - 2^m - 2^{m+i}|. \tag{6.11}$$

Suppose that $x_3 \geq m+i+2$. From (6.11), it is clear that $g(\mathbf{x}) \geq 2^m$.

Suppose that $x_3 = m+i+1$. Then $x_1 + x_2 = 2m+j-2$, and so $x_2 \geq m$ or $x_1 = x_2 = m-1$. By (6.11), we have $g(\mathbf{x}) \geq 2^m$.

Suppose that $x_3 = m+i$. Clearly, $x_1 + x_2 = 2m+j-1$. This implies that $x_2 \geq m+1$ or $x_1 = m-1$, $x_2 = m$ and $i = j = k = 0$. Thus, from (6.11), we have $g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m| \geq 2^m$, or $g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m| = 2^{m-1}$ and $x_1 = m-1$, $x_2 = m$, $x_3 = m$ and $x_4 = m+1$, which is a special case of (iii).

Suppose that $x_3 = m+i-1$. Note that $x_1 \leq x_2 \leq x_3 = m+i-1$. Clearly, $x_1 + x_2 = 2m + j$. By (6.11), it follows immediately that

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i-1}|. \tag{6.12}$$

In (6.12), if $x_2 = m+i-1$, then $x_1 = m+j-i+1 \geq m+1$. Thus $g(\mathbf{x}) = |2^{x_1} - 2^m| \geq 2^m$; if $x_2 \leq m+i-2$, then, as $x_1 \leq x_2$ and by (6.12) we have $g(\mathbf{x}) \geq 2^m$.

Suppose that $x_3 \leq m+i-2$. Note that $x_1 \leq x_2 \leq x_3 = m+i-2$. Clearly, $2^{m+i} \geq 4 \times 2^{x_3}$. According to (6.11), it follows immediately that $g(\mathbf{x}) \geq 2^m$.

Subcase 3.2. $j \leq k-1$. Since $x_4 = m+k+1$, by (6.10), we have

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} + 2 \times 2^{m+k-1} - 2^m - 2^{m+i} - 2^{m+j}|. \tag{6.13}$$

Recalling that $x_1 + x_2 + x_3 = 3m + i + j - 1$, we see that $x_3 \geq m+1$ or $x_3 = x_2 = m$. Hence, it is easy to see from (6.13) that $g(\mathbf{x}) \geq 2^m$.

Case 4. $x_4 = m+k$. Then $x_1 + x_2 + x_3 = 3m + i + j$ and

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} - 2^m - 2^{m+i} - 2^{m+j}|. \tag{6.14}$$

By Theorem 6.4.1 and (6.14), we get (i) and part one of (iii) of the theorem.

Case 5. $x_4 = m+k-1$. Then $x_1 + x_2 + x_3 = 3m + i + j + 1$.

Subcase 5.1. $x_3 = x_4 = m+k-1$. Then $x_1 + x_2 = 2m + i + j + 2 - k$. From (6.10), we have

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i} - 2^{m+j}|. \tag{6.15}$$

We consider (6.15) by distinguishing the following subcases.

Subcase 5.1.1. $x_2 \geq m+j+2$. It is clear that $g(\mathbf{x}) \geq 2^m$.

Subcase 5.1.2. $x_2 = m+j+1$. Then $x_1 = m+i+1-k$. By (6.15), it follows that

$$g(\mathbf{x}) = |2^{x_1} + 2^{m+j} - 2^m - 2^{m+i}|. \tag{6.16}$$

Subcase 5.1.2.1. Suppose that $i = j$. From (6.16), we have

$$g(\mathbf{x}) = |2^{x_1} - 2^m|. \tag{6.17}$$

In (6.17), if $x_1 \geq m + 1$, then $g(\mathbf{x}) \geq 2^m$; if $x_1 = m$, then $i = k - 1$ and $x_2 = m + j + 1 = m + i + 1 = m + k > x_4$, which contradicts the assumption $x_2 \leq x_4$; if $x_1 = m - 1$, then $i = j = k - 2$, and thus $x_2 = x_3 = x_4 = m + k - 1$ and $g(\mathbf{x}) = 2^{m-1}$, which is part two of (iii); if $x_1 \leq m - 2$, we see that $2^{m-1} < g(\mathbf{x}) < 2^m$. Assume that $x_1 = m - b$. Then $x_2 = m + i + 1$, $x_3 = x_4 = m + b + i$, $k = b + i + 1$ and $i = j$, where b is a positive integer and $b \geq 2$, which implies (iv).

Subcase 5.1.2.2. Suppose that $i = j - 1$. Clearly, $j = i + 1 \geq 1$ and $x_1 = m + i + 1 - k$. From (6.16), it is not difficult to see that

$$g(\mathbf{x}) = |2^{x_1} + 2^{m+j-1} - 2^m|. \tag{6.18}$$

In (6.18), if $i \geq 1$, then $j \geq 2$, and it follows immediately that $g(\mathbf{x}) = 2^m$; if $i = 0$, then $j = 1$ and $x_2 = m + j + 1 \leq x_3 = m + k - 1$, thus $k \geq 3$. So, if $i = 0$, from (6.18) we have $j = 1$, $k \geq 3$ and $0 < g(\mathbf{x}) = 2^{m-k+1} \leq 2^{m-2}$, which is (ii).

Subcase 5.1.2.3. Suppose that $i \leq j - 2$. Since $2^{m+j} = 4 \times 2^{m+j-2} \geq 4 \times 2^{m+i}$, it follows from (6.16) that $g(\mathbf{x}) > 2^m$.

Subcase 5.1.3. $x_2 = m + j$. So, $x_1 = m + i + 2 - k$. From (6.15), it is clear that

$$g(\mathbf{x}) = |2^{x_1} - 2^m - 2^{m+i}|. \tag{6.19}$$

In (6.19), if $k \geq 2$, then $x_1 \leq m + i$, and so $g(\mathbf{x}) \geq 2^m$; if $k \leq 1$, then $x_1 \geq m + i + 1 > x_4 = m + k - 1$, which contradicts our assumption $x_1 \leq x_4$.

Subcase 5.1.4. $x_2 \leq m + j - 1$. As $x_1 \leq x_2$, from (6.15), it is clear that $g(\mathbf{x}) > 2^m$.

Subcase 5.2. $x_4 = m + k - 1$ and $x_3 = m + k - 2$. From (6.10), we have

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i} - 2^{m+j} - 2^{m+k-2}|. \tag{6.20}$$

Subcase 5.2.1. Suppose that $x_2 = x_3 = m+k-2$. So, $x_1 = m+i+j-2k+5 = m+j-(k-i)-(k-5)$. From (6.20), it follows that

$$g(\mathbf{x}) = |2^{x_1} - 2^m - 2^{m+i} - 2^{m+j}|. \tag{6.21}$$

In (6.21), if $k \geq 5$, then $m+j \geq x_1$, and thus $g(\mathbf{x}) > 2^m$; if $k \leq 4$, then $x_1 \leq x_2 \leq m+2$. So, when $x_1 \leq m+1$, from (6.21) we have $g(\mathbf{x}) \geq 2^m$; when $x_1 = m+2$, we must have that $k=4$ and $i+j=5$ (otherwise if $k \leq 3$, then $x_1 \leq x_2 = m+1$), which implies that $j \geq 3$. From (6.21), it is clear that $g(\mathbf{x}) > 2^m$.

Subcase 5.2.2. Suppose that $x_2 \leq x_3 - 1 = m+k-3$. As $x_1 \leq x_2$, from (6.20) we have

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} - 2^m - 2^{m+i} - 2^{m+j} - 2 \times 2^{m+k-3}| > 2^m.$$

Subcase 5.3. $x_4 = m+k-1$ and $x_3 \leq m+k-3$. By (6.10), we know that

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} - 2^m - 2^{m+i} - 2^{m+j} - 4 \times 2^{m+k-3}| > 2^m.$$

Case 6. $x_4 \leq m+k-2$. From (6.10), it is not difficult to deduce that

$$g(\mathbf{x}) \quad = \quad |2^{x_1} + 2^{x_2} + 2^{x_3} + 2^{x_4} - 2^m - 2^{m+i} - 2^{m+j} - 4 \times 2^{m+k-2}| > 2^m.$$

This completes the proof of the theorem. $\square$

Theorem 6.5.2. *Let* $G = K(n_1, n_2, n_3, n_4)$ *and* $S \subset E(G)$ *such that* $n = n_1+n_2+n_3+n_4$ *and* $|S| = s \geq 1$. *If* $n > \sqrt{3\sum_{1\leq i<j\leq 4}(n_i-n_j)^2 + 24s + 4q(T_{n,4}) - 4q(G) + 4s + 4}$, *then* $\mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$ *is* χ*-closed.*

Proof. Let $G = K(n_1, n_2, n_3, n_4)$ and $H \in \mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$. Then there exists a subset S of $E(G)$ such that $H = G - S$. Let Y be a graph such that $Y \sim H$. We prove that $Y \in \mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$.

By Lemma 6.2.2, there exists a graph $F = K(m_1, m_2, m_3, m_4)$ and $S' \subseteq E(F)$ such that $Y = F - S'$, $|S'| = s'$, $s' = q(F) - q(G) + s$ and $\sum_{i=1}^{4} n_i = \sum_{i=1}^{4} m_i$. Without loss of generality, assume that $n_1 \leq n_2 \leq n_3 \leq n_4$ and $m_1 \leq m_2 \leq m_3 \leq m_4$. From the condition of the theorem, we have

$$\frac{n - \sqrt{3\sum_{1\leq i<j\leq t}(n_i-n_j)^2 + 24s}}{4} > q(T_{n,4}) - q(G) + s + 1.$$

By Lemmas 5.2.4 and 6.2.2, we have that $n_1 > q(T_{n,4}) - q(G) + s + 1 \geq s + 2$ and $m_1 > q(T_{n,4}) - q(G) + s + 1 \geq s' + 2$. So, from Lemma 5.2.7, it follows that

$$\alpha(H, 5) = \alpha(G, 5) + \alpha_5(H), \quad s \leq \alpha_5(H) \leq 2^s - 1$$

and

$$\alpha(Y, 5) = \alpha(F, 5) + \alpha_5(Y), \quad s' \leq \alpha_5(Y) \leq 2^{s'} - 1.$$

Suppose that $n_4 - n_1 = \theta$ and

$$g(\mathbf{x}) = |2^{x_1} + 2^{x_2} + 2^{x_3} + 2^{x_4} - 2^{n_1} - 2^{n_2} - 2^{n_3} - 2^{n_4}|.$$

By Lemma 5.2.5, $\alpha(H, 5) = \alpha(Y, 5)$. So, by Lemma 5.2.3, we have

$$\alpha(Y, 5) - \alpha(H, 5) = \sum_{i=1}^{4} 2^{m_i - 1} - \sum_{i=1}^{4} 2^{n_i - 1} + \alpha_5(Y) - \alpha_5(H)$$

$$= \begin{cases} \frac{1}{2} g(m_1, m_2, m_3, m_4) + \alpha_5(Y) - \alpha_5(H), & \text{if } \alpha(F, 5) \geq \alpha(G, 5), \\ -\frac{1}{2} g(m_1, m_2, m_3, m_4) + \alpha_5(Y) - \alpha_5(H), & \text{if } \alpha(F, 5) < \alpha(G, 5). \end{cases} \tag{6.22}$$

By Theorem 6.5.1, we need only consider the following cases.

Case 1. $(m_1, m_2, m_3, m_4) = (n_1, n_2, n_3, n_4)$. Then $F = G$ and $s = s'$; i.e. $Y \in \mathcal{G}^{-s}_{n_1, n_2, n_3, n_4}$.

Case 2. $(m_1, m_2, m_3, m_4) = (n_1 - \theta + 1, n_1 + 2, n_1 + \theta - 1, n_1 + \theta - 1)$ and $\theta \geq 3$.

From Theorem 6.5.1(ii), we have $n_1 = n_2$, $n_3 = n_1 + 1$, $n_4 = n_1 + \theta$ and $\theta \geq 3$. It is not difficult to verify that $\alpha(F, 5) > \alpha(G, 5)$. Thus, by (6.22) and Theorem 6.5.1(ii), we have

$$\alpha(Y, 5) - \alpha(H, 5) = 2^{n_1 - \theta} + \alpha_5(Y) - \alpha_5(H). \tag{6.23}$$

By Lemma 5.2.4, $n_1 > q(T_{n,4}) - q(G) + s + 1 \geq \theta + s$; i.e., $s < n_1 - \theta$. Since $s \leq \alpha_5(H) \leq 2^s - 1$ and $0 \leq s' \leq \alpha_5(Y) \leq 2^{s'} - 1$, by (6.23) we have

$$\alpha(Y, 5) - \alpha(H, 5) > 2^s + \alpha_5(Y) - 2^s + 1 \geq 1.$$

This contradicts $\alpha(Y, 5) = \alpha(H, 5)$.

Case 3. $(m_1, m_2, m_3, m_4) = (n_1 - 1, n_1 + 1, n_3, n_4)$.

From part one of Theorem 6.5.1 (iii), it is clear that $n_1 = n_2$, $\alpha(F,5) \geq \alpha(G,5)$ and $g(m_1,m_2,m_3,m_4) = 2^{n_1-1}$. Hence, from (6.22), it follows that

$$\alpha(Y,5) - \alpha(H,5) = 2^{n_1-2} + \alpha_5(Y) - \alpha_5(H). \tag{6.24}$$

Recalling that $n_1 \geq s+2$, $s \leq \alpha_5(H) \leq 2^s - 1$ and $0 \leq s' \leq \alpha_5(Y) \leq 2^{s'} - 1$, we obtain, from (6.24), that

$$\alpha(Y,5) - \alpha(H,5) \geq 2^s + \alpha_5(Y) - 2^s + 1 \geq 1,$$

which again contradicts $\alpha(Y,5) = \alpha(H,5)$.

Case 4. $(m_1,m_2,m_3,m_4) = (n_1-1, n_1+\theta-1, n_1+\theta-1, n_1+\theta-1)$.

By part two of Theorem 6.5.1 (iii), $n_2 = n_3 = n_4 - 2$ and $\theta \geq 2$. In this case, it is not hard to deduce that $\alpha(F,5) < \alpha(G,5)$ and $g(m_1,m_2,m_3,m_4) = 2^{n_1-1}$. Hence, from (6.22), it follows that

$$\alpha(Y,5) - \alpha(H,5) = -2^{n_1-2} + \alpha_5(Y) - \alpha_5(H). \tag{6.25}$$

Recalling that $q(F) - q(G) + s = s'$ and $q(T_{n,4}) - q(F) \geq \theta - 1$, by Lemma 5.2.4, we have $n_1 > q(T_{n,4}) - q(G) + s + 1 = q(T_{n,4}) - q(F) + q(F) - q(G) + s + 1 \geq s' + \theta \geq s' + 2$. Note that $0 \leq s \leq \alpha_5(H) \leq 2^s - 1$ and $0 \leq s' \leq \alpha_5(Y) \leq 2^{s'} - 1$. It thus follows, from (6.25), that

$$\alpha(Y,5) - \alpha(H,5) \leq -2^{s'} + 2^{s'} - 1 - \alpha_5(H) \leq -1,$$

which again contradicts the fact that $\alpha(Y,5) = \alpha(H,5)$.

Case 5. $(m_1,m_2,m_3,m_4) = (n_1-b, n_1+i+1, n_1+i+b, n_1+i+b)$.

By Theorem 6.5.1 (iv), $n_2 = n_3 = n_1 + i, n_4 = n_1 + b + i + 1$ and $b \geq 2$. So, we deduce that $\alpha(F,5) < \alpha(G,5)$ and $g(m_1,m_2,m_3,m_4) > 2^{n_1-1}$. Hence, from (6.22), it follows that

$$\alpha(Y,5) - \alpha(H,5) \leq -2^{n_1-2} + \alpha_5(Y) - \alpha_5(H). \tag{6.26}$$

By Lemma 5.2.4, $q(T_{n,4}) - q(F) \geq 2b + i - 1 \geq 3$. As $q(F) - q(G) + s = s'$, $n_1 > q(T_{n,4}) - q(F) + q(F) - q(G) + s + 1 \geq s' + 2$. Note that $0 \leq s \leq \alpha_5(H) \leq 2^s - 1$ and $0 \leq s' \leq \alpha_5(Y) \leq 2^{s'} - 1$. It thus follows, from (6.26), that

$$\alpha(Y,5) - \alpha(H,5) < -2^{s'} + 2^{s'} - 1 - \alpha_5(H) \leq -1,$$

which again contradicts the fact that $\alpha(Y,5)=\alpha(H,5)$.

Case 6. (m_1,m_2,m_3,m_4) does not take the values from Case 1 to Case 5.

By Theorem 6.5.1, $g(m_1,m_2,m_3,m_4)\geq 2^{n_1}$. From (6.22), it follows that

$$\alpha(Y,5)-\alpha(H,5)\geq 2^{n_1-1}+\alpha_5(Y)-\alpha_5(H) \tag{6.27}$$

or

$$\alpha(Y,5)-\alpha(H,5)\leq -2^{n_1-1}+\alpha_5(Y)-\alpha_5(H) \tag{6.28}$$

As $s\leq n_1-2$ and $s'\leq n_1-2$, by (6.27) and (6.28) we have

$$\alpha(Y,5)-\alpha(H,5)\geq 2^{n_1-1}+\alpha_5(Y)-\alpha_5(H)>2^s+\alpha_5(Y)-2^s+1\geq 1$$

or

$$\alpha(Y,5)-\alpha(H,5)\leq -2^{n_1-1}+\alpha_5(Y)-\alpha_5(H)<-2^{s'}+2^{s'}-1-\alpha_5(H)\leq -1.$$

This contradicts the fact that $\alpha(F,5)=\alpha(H,5)$.

We thus conclude from the above arguments that $(m_1,m_2,m_3,m_4)=(n_1,n_2,n_3,n_4)$. Clearly, $F=G$ and $|S|=|S'|$. Hence $Y\in\mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$, as required. □

The following lemma follows from the proof of Theorem 5.5.4.

Lemma 6.5.1. *Let $G=K(n_1,n_2,n_3,n_4)$ with n vertices. Then*

$$q(T_{n,4})-q(G)\leq \sum_{1\leq i<j\leq 4}\frac{(n_i-n_j)^2}{8}.$$

From Lemma 6.5.1 and Theorem 6.5.2, the following corollary follows immediately.

Corollary 6.5.1. *Let $G=K(n_1,n_2,n_3,n_4)$ and $s\geq 1$.*
(i) If $min\{n_1,n_2,n_3,n_4\}>\sum_{1\leq i<j\leq 4}\frac{(n_j-n_i)^2}{8}+\frac{\sqrt{3\sum_{1\leq i<j\leq 4}(n_i-n_j)^2+24s}}{4}+s+1$, then $\mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$ is χ-closed, where $i,j=1,2,3,4$.
(ii) If $m>k^2/2+\sqrt{3k^2+6s}/2+s+1$, then $\mathcal{G}^{-s}_{m,m+i,m+j,m+k}$ is χ-closed.

Theorem 6.5.3. *Let $G = K(n_1, n_2, n_3, n_4)$ and $n = n_1+n_2+n_3+n_4$. Suppose that $n_1 \leq n_2 \leq n_3 \leq n_4$, $s \geq 1$ and $n > \sqrt{3\sum_{1\leq i<j\leq 4}(n_i - n_j)^2 + 24s + 4q(T_{n,4}) - 4q(G) + 4s + 4}$. Then*

(i) every $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique, for any (i,j) if $n_2+n_3 \neq n_1+n_4$, where $i \neq j$ and $i, j = 1, 2, 3, 4$;

(ii) every $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique, for any (i,j) if $n_1 = n_2$ and $n_3 = n_4$, where $i \neq j$ and $i, j = 1, 2, 3, 4$;

(iii) every $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique, for $(i,j) \in \{(1,2), (2,1), (1,3), (3,1), (2,4), (4,2), (3,4), (4,3)\}$.

Proof. Denote by X the family $\{K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4) | i \neq j$ and $i, j = 1, 2, 3, 4\}$. Assume that $F \in X$. Let H be a graph such that $H \sim F$. It suffices to prove that $H = F$.

By Theorem 6.5.2, $H \in \mathcal{G}^{-s}_{n_1,n_2,n_3,n_4}$. Since $\alpha_5(H) = \alpha(F) = 2^s - 1$, by Lemma 5.2.7 we have $H \in X$. In the following, we consider the numbers of triangles of H and F. Let M be the number of triangles of $K(n_1, n_2, n_3, n_4)$ and $N_A(i,j)$ the number of triangles in the graph $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$. Then the following results can be obtained easily.

$$\begin{aligned}
N_A(1,2) = N_A(2,1) &= M - s(n_3 + n_4),\\
N_A(1,3) = N_A(3,1) &= M - s(n_2 + n_4),\\
N_A(1,4) = N_A(4,1) &= M - s(n_2 + n_3),\\
N_A(2,3) = N_A(3,2) &= M - s(n_1 + n_4),\\
N_A(2,4) = N_A(4,2) &= M - s(n_1 + n_3),\\
N_A(3,4) = N_A(4,3) &= M - s(n_1 + n_2).
\end{aligned} \tag{6.29}$$

Since $n_1 \leq n_2 \leq n_3 \leq n_4$, it is not difficult to see that

$$n_1 + n_2 \leq n_1 + n_3 \leq n_1 + n_4 = n_2 + n_3 \leq n_2 + n_4 \leq n_3 + n_4 \tag{6.30}$$

or

$$n_1 + n_2 \leq n_1 + n_3 \leq n_1 + n_4 < n_2 + n_3 \leq n_2 + n_4 \leq n_3 + n_4 \tag{6.31}$$

or

$$n_1 + n_2 \leq n_1 + n_3 \leq n_2 + n_3 < n_1 + n_4 \leq n_2 + n_4 \leq n_3 + n_4. \tag{6.32}$$

We partition the family X into the following classes:
$\mathcal{G}_1 = \{K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$,
$\mathcal{G}_2 = \{K_{1,3}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{3,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$,
$\mathcal{G}_3 = \{K_{1,4}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{4,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$,
$\mathcal{G}_4 = \{K_{2,3}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{3,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$,
$\mathcal{G}_5 = \{K_{2,4}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{4,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$,
$\mathcal{G}_6 = \{K_{3,4}^{-K_{1,s}}(n_1, n_2, n_3, n_4), K_{4,3}^{-K_{1,s}}(n_1, n_2, n_3, n_4)\}$.

Obviously, $X = \cup_{i=1}^{6} \mathcal{G}_i$. Note that if $n_i = n_j$, then, for any $l \neq i, j$, we have

$$K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4),$$

and

$$K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4).$$

Since $H \sim F$, $N_A(H) = N_A(F)$. Hence, if there exist two distinct positive integers a and b such that $F \in \mathcal{G}_a$ and $H \in \mathcal{G}_b$, then by $N_A(H) = N_A(F)$ and (6.29)-(6.32), the following results can be shown:

(i) if $n_1 + n_4 \neq n_2 + n_3$, then $\mathcal{G}_a = \mathcal{G}_b$;

(ii) if $n_1 + n_4 = n_2 + n_3$ and $n_1 = n_2$, then $\mathcal{G}_a = \mathcal{G}_b$;

(iii) if $n_1 + n_4 = n_2 + n_3$ and $n_1 \neq n_2$, then $\mathcal{G}_a = \mathcal{G}_b$ if and only if $a \neq 3, 4$ or $b \neq 3, 4$.

So, by the conditions of (i), (ii) and (iii) of the theorem, it suffices to prove that $F = H$ if F and H belong to the same set $\mathcal{G}_a$. Without loss of generality, assume that $F \in \mathcal{G}_1$ and $H \in \mathcal{G}_1$. Take $F = K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$. Clearly, $H = K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ or $H = K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$. So, we need only prove that if $K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4) \neq K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$, then

$$P(K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4), \lambda).$$

Note that $K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4) = K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ for $n_1 = n_2$. By Lemma 5.2.1, for $n_1 \neq n_2$, we have

$$\sigma(K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4), x) = \sigma(K^{-K_{1,s}}(A_1, A_2), x)\sigma(K_{n_3}, x)\sigma(K_{n_4}, x)$$

and

$$\sigma(K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4), x) = \sigma(K^{-K_{1,s}}(A_2, A_1), x)\sigma(K_{n_3}, x)\sigma(K_{n_4}, x).$$

From the proof of Theorem 6.5.2, we have $min\{n_1, n_2, n_3, n_4\} \geq s + 2$. By Lemma 6.2.1 we know that if $n_1 \neq n_2$, then

$$\sigma(K^{-K_{1,s}}(A_1, A_2), x) \neq \sigma(K^{-K_{1,s}}(A_2, A_1), x).$$

So,

$$\sigma(K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4), x) \neq \sigma(K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4), x),$$

which implies that

$$P(K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4), \lambda).$$

Hence $H = F$ and both $K_{1,2}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ and $K_{2,1}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ are χ-unique.

Similarly, we can show that $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$ is χ-unique for other pairs (i, j) satisfying one of the conditions (i), (ii) and (iii) of the theorem. $\square$

From Lemma 6.5.1 and Theorem 6.5.3, we have

Corollary 6.5.2. *Let* $n_1 \leq n_2 \leq n_3 \leq n_4$ *and* $s \geq 1$.

(i) Assume that $n_1 > \sum\limits_{1 \leq i < j \leq 4} \frac{(n_j - n_i)^2}{8} + \frac{\sqrt{3 \sum_{1 \leq i < j \leq 4} (n_i - n_j)^2 + 24s}}{4} + s + 1$. *If* $n_1 + n_4 \neq n_2 + n_3$, *or* $n_1 = n_2$ *and* $n_3 = n_4$, *then every* $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$, *for* $i \neq j$ *and* $i, j = 1, 2, 3, 4$, *is* χ*-unique.*

(ii) Assume that $n_1 > k^2/2 + \sqrt{3k^2 + 6s}/2 + s + 1$ *and* $n_4 - n_1 = k$. *If* $n_1 + n_4 \neq n_2 + n_3$, *or* $n_1 = n_2$ *and* $n_3 = n_4$, *then* $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4)$, *for* $i \neq j$ *and* $i, j = 1, 2, 3, 4$, *is* χ*-unique.*

Theorem 6.5.4. *Let* $G = K(n_1, n_2, n_3, n_4)$ *with* $n_1 \leq n_2 < n_3 \leq n_4$ *and let* $s \geq 1$. *If* $n > \sqrt{3 \sum_{1 \leq i < j \leq 4} (n_i - n_j)^2 + 24s + 4q(T_{n,4}) - 4q(G) + 4s + 4}$, *then* $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$ *is* χ*-unique, where* $n = n_1 + n_2 + n_3 + n_4$.

Proof. Suppose that $H \sim K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$. By Theorem 6.5.2 and Lemma 5.2.7, $H \in \mathcal{G}_{n_1,n_2,n_3,n_4}^{-s}$ and $\alpha_5(H) = s$. Next we consider the number of triangles of H. Without loss of generality, assume that $S \subset E(G)$ and $H = G - S$. Let $e \in S$. Denote by $N_A(e)$ the number of triangles in G containing the edge e. Then $N_A(e) \leq n_3 + n_4$. As $n_1 \leq n_2 < n_3 \leq n_4$, we know that $N_A(e) = n_3 + n_4$ if and only if e is an edge of the subgraph $K(A_1, A_2)$. So,

$$N_A(H) \geq N_A(G) - s(n_3 + n_4),$$

where equality holds if and only if e is an edge of the subgraph $K(A_1, A_2)$ in G. Recalling $\alpha_5(H) = s$, by Lemma 5.2.7 we have $H = K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$. □

Similarly, from Lemma 6.5.1 and Theorem 6.5.4, we have

Corollary 6.5.3. *Suppose that $n_1 \leq n_2 < n_3 \leq n_4$ and $s \geq 1$.*
(i) If $min\{n_1, n_2, n_3, n_4\} > \sum\limits_{1\leq i<j\leq 4} \frac{(n_j-n_i)^2}{8} + \frac{\sqrt{3\sum_{1\leq i<j\leq 4}(n_i-n_j)^2+24s}}{4} + s + 1$, then $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$ is χ-unique.
(ii) If $n_1 > k^2/2 + \sqrt{3k^2+6s}/2 + s + 1$ and $n_4 - n_1 = k$, then $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4)$ is χ-unique.

6.6 The chromaticity of t-partite graphs

For $t \geq 5$, it is very difficult to study the chromaticity of t-partite graphs by using the same procedure as in the former sections. So, in this section, we investigate the chromaticity of t-partite graphs obtained by deleting some edges from a complete t-partite graph $K(n, n, \cdots, n, n+1, n+1, \cdots, n+1)$, where $t \geq 5$.

Let x_i be positive integers, for all $i = 1, 2, \cdots, t$. First, we give the lower bounds the function defined as

$$\varphi(\mathbf{x}) = \sum_{i=1}^{t} 2^{x_i}, \tag{6.33}$$

where $\sum\limits_{i=1}^{t} x_i = t_1 n + t_2(n+1)$, $t_1 \geq 1$ and $t_1 + t_2 = t$, $\mathbf{x} = (x_1, x_2, \cdots, x_t)$.

Lemma 6.6.1. *Let $\mathbf{x_1} = (x_1, x_2, \cdots, x_i, \cdots, x_j, \cdots, x_t)$ and $\mathbf{x_2} = (x_1, x_2, \cdots, x_i+1, \cdots, x_j-1, \cdots, x_t)$. If $x_j \geq x_i + 2$, then*

$$\varphi(\mathbf{x_1}) - \varphi(\mathbf{x_2}) \geq 2^{x_j - 2}.$$

Proof. By (6.33) and $x_j \geq x_i + 2$, we can check this directly. $\square$

Let $\mathbf{x}' = (x_1, x_2, \cdots, x_i, \cdots, x_j, \cdots, x_t)$ and $\mathbf{x}'' = (x_1, x_2, \cdots, x_i + 1, \cdots, x_j - 1, \cdots, x_t)$. If $i < j$ and $x_i + 2 \leq x_j$, then $\mathbf{x}''$ is said to be *an improvement* of $\mathbf{x}'$. By Lemma 6.6.1, one sees that $\varphi(\mathbf{x}') > \varphi(\mathbf{x}'')$ if $\mathbf{x}''$ is an improvement of $\mathbf{x}'$. Let $\Pi = t_1 2^n + t_2 2^{n+1}$, where $t_1 \geq 1$ and $t_1 + t_2 = t$. For convenience, we replace $\overbrace{n, n, \cdots, n}^{a}$ by $a \times n$. For example, $K(t_1 \times n, t_2 \times (n+1))$ denotes the graph $K(\overbrace{n, n, \cdots, n}^{t_1}, \overbrace{n+1, n+1, \cdots, n+1}^{t_2})$ and $\mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1))$ denotes the family $\{K(\overbrace{n, n, \cdots, n}^{t_1}, \overbrace{n+1, n+1, \cdots, n+1}^{t_2}) - s | s \geq 1\}$.

Theorem 6.6.1. *Let $\varphi(\mathbf{x})$ be the function defined by (6.33). Then $\varphi(\mathbf{x}) \geq \Pi + 2^{n-1}$ except for $\varphi(\mathbf{x}) = \Pi$ if $\mathbf{x} = (t_1 \times n, t_2 \times n + 1)$.*

Proof. Suppose that $\mathbf{x} = (x_1, x_2, \cdots, x_t)$ such that $\sum_{i=1}^{t} x_i = t_1 n + t_2(n+1)$ and $t_1 + t_2 = t$. Let $\mathbf{x_1} = \mathbf{x}$. We construct a sequence $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \cdots, \mathbf{x_m}$ from $\mathbf{x_1}$ such that $|x_i^m - x_j^m| \leq 1$ for any two elements x_i^m and x_j^m in $\mathbf{x_m}$ and $\mathbf{x_{k+1}}$ is an improvement of $\mathbf{x_k}$, where $\mathbf{x_k} = (x_1^k, x_2^k, \cdots, x_i^k, \cdots, x_j^k, \cdots, x_t^k)$ and $k = 1, 2, \cdots, m-1$.

From Lemma 6.6.1, we have that

$$\varphi(\mathbf{x_k}) - \varphi(\mathbf{x_{k+1}}) \geq 2^{x_j^k - 2}, \tag{6.34}$$

for some j.

So, from (6.34), we have that $\varphi(\mathbf{x}) \geq \Pi$ and the equality holds if and only if $\mathbf{x} = \mathbf{x_m} = (n, n, \cdots, n, n+1, n+1, \cdots, n+1)$. Note that $\mathbf{x_m} = (t_1 \times n, t_2 \times (n+1))$ and $\mathbf{x_m}$ is an improvement of $\mathbf{x_{m-1}}$. So, $\mathbf{x_{m-1}}$ must be one of the following cases.

Case 1. $\mathbf{x_{m-1}} = (n-1, (t_1 - 2) \times n, (t_2 + 1) \times (n+1))$. So,

$$\varphi(\mathbf{x_{m-1}}) = 2^{n-1} + (t_1 - 2)2^n + (t_2 + 1)2^{n+1} = \Pi + 2^{n-1}.$$

Case 2. $\mathbf{x_{m-1}} = ((t_1 + 1) \times n, (t_2 - 2) \times (n+1))$. Then

$$\varphi(\mathbf{x_{m-1}}) = (t_1 + 1)2^n + (t_2 - 2)2^{n+1} + 2^{n+2} = \Pi + 2^n.$$

Case 3. $\mathbf{x_{m-1}} = (n-1, (t_1-1)\times n, (t_2-1)\times(n+1), n+2)$. We have

$$\varphi(\mathbf{x_{m-1}}) = 2^{n-1} + (t_1-1)2^n + (t_2-1)2^{n+1} + 2^{n+2} = \Pi + 2^n + 2^{n-1}.$$

Let $\mathcal{X}_k = \{\mathbf{x_k} | \mathbf{x_{k+1}}$ is an improvement of $\mathbf{x_k}\}$, where $k = 1, 2, \cdots, m-1$ and $\mathcal{X}_m = \{(t_1\times n, t_2\times(n+1))\}$. From (6.34), we have

$$min\{\varphi(\mathbf{x_k}) | \mathbf{x_k} \in \mathcal{X}_\mathbf{k}\} > min\{\varphi(\mathbf{x_{k+1}}) | \mathbf{x_{k+1}} \in \mathcal{X}_\mathbf{k+1}\}.$$

From the above arguments we have that the theorem holds. □

Theorem 6.6.2. *Let $s \geq 1$, $n \geq 2$ and $t_1 \geq 1$. If $n \geq s+2$, then $\mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1))$ is χ-closed.*

Proof. Let $H \in \mathcal{G}^{-s}(t_1\times n, t_2\times(n+1))$. Then there exists a subset S of $E(K(t_1\times n, t_2\times(n+1)))$ such that $H = K(t_1\times n, t_2\times(n+1)) - S$. Let Y be a graph such that $Y \sim H$. The definition of χ-closed implies that it is sufficient to prove that $Y \in \mathcal{G}^{-s}(t_1\times n, t_2\times(n+1))$.

By Lemma 6.2.2, we know that there exists a graph $F = K(m_1, m_2, \cdots, m_t)$ and $S' \subset E(F)$ such that $Y = F - S'$ and $|S'| = s'$, $s' = q(F) - q(G) + s$ and $\sum_{i=1}^{t} n_i = \sum_{i=1}^{t} m_i$. Note that

$$\alpha(H, t+1) = 2^{-1}\Pi - t + \alpha_{t+1}(H)$$

and

$$\alpha(Y, t+1) = 2^{-1}\sum_{i=1}^{t} 2^{m_i} - t + \alpha_{t+1}(Y).$$

Without loss of generality, assume that $m_1 \leq m_2 \leq \cdots \leq m_t$ and

$$\psi(\mathbf{x}) = \sum_{i=1}^{t} 2^{m_i} - \Pi.$$

By Lemma 5.2.5, we have that $\alpha(H, t+1) = \alpha(Y, t+1)$. So,

$$\alpha(Y, t+1) - \alpha(H, t+1) = 2^{-1}\psi(\mathbf{x}) + \alpha_{t+1}(Y) - \alpha_{t+1}(H). \tag{6.35}$$

From Theorem 6.6.1, it suffices to consider the following cases.

Case 1. $(m_1, m_2, \cdots, m_t) = (t_1 \times n, t_2 \times (n+1))$. Then $F = K(t_1 \times n, t_2 \times (n+1))$ and $s = s'$, i.e., $Y \in \mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1))$.

Case 2. $(m_1, m_2, \cdots, m_t)$ does not take the values of Case 1.

By Theorem 6.6.1, it is clear that $\psi(\mathbf{x}) \geq 2^{n-1}$. From (6.35), it follows that

$$\alpha(Y, t+1) - \alpha(H, t+1) \geq 2^{n-2} + \alpha_{t+1}(Y) - \alpha_{t+1}(H). \tag{6.36}$$

Since $n \geq s+2$, by Lemma 5.2.7, we have that $s \leq \alpha_{t+1}(H) \leq 2^s - 1$ and $s' \leq \alpha_{t+1}(Y)$. Thus, by (6.36) we have that

$$\begin{aligned} \alpha(Y, t+1) - \alpha(H, t+1) &\geq 2^{n-2} + \alpha_{t+1}(Y) - \alpha_{t+1}(H) \\ &\geq 2^s + \alpha_{t+1}(Y) - 2^s + 1 \geq 1, \end{aligned}$$

This contradicts $\alpha(F, t+1) = \alpha(H, t+1)$.

From the above arguments, we have that $Y \in \mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1)))$. $\square$

Suppose that $K(t_1 \times n, t_2 \times (n+1))$ has t partition sets A_i such that $|A_i| = n$, for $1 \leq i \leq t_1$, and $|A_i| = n+1$, for $t_1 + 1 \leq i \leq t$. Note that if $n_i = n_j$, then, for any $l \neq i, j$, we can see that

$$K_{i,l}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1)) = K_{j,l}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1)),$$

and

$$K_{l,i}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1)) = K_{l,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1)).$$

So, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$, for $i \neq j$ and $i, j = 1, 2, \cdots, t$, include the following different graphs: $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$, for $|A_i| = |A_j| = n$, denoted by $H^{-K_{1,s}}(n, n)$, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$, for $|A_i| = |A_j| = n+1$, denoted by $H^{-K_{1,s}}(n+1, n+1)$, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$, for $|A_i| = n$ and $|A_j| = n+1$, denoted by $H^{-K_{1,s}}(n, n+1)$, $K_{i,j}^{-K_{1,s}}(t_1 \times n, t_2 \times (n+1))$, for $|A_i| = n+1$ and $|A_j| = n$, denoted by $H^{-K_{1,s}}(n+1, n)$. Let $\mathcal{H}^{-K_{1,s}} = \{H^{-K_{1,s}}(n, n), H^{-K_{1,s}}(n+1, n+1), H^{-K_{1,s}}(n, n+1), H^{-K_{1,s}}(n+1, n)\}$.

Theorem 6.6.3. *Let $G \in \mathcal{H}^{-K_{1,s}}$. If $s \geq 1$ and $n \geq s+2$, then G is χ-unique.*

Proof. Suppose that $F \in \mathcal{H}^{-K_{1,s}}$ and $H \sim F$. From Theorem 6.6.2,

$$H \in \mathcal{G}^{-s}(t_1 \times n, t_2 \times (n+1)).$$

Note that $\alpha_{t+1}(H) = \alpha_{t+1}(F) = 2^s - 1$. By Lemma 5.2.7, we know that

$$H \in \mathcal{H}^{-K_{1,s}}.$$

So, we now prove that $P(G_1, \lambda) \neq P(G_2, \lambda)$ for G_1 and G_2 in $\mathcal{H}^{-K_{1,s}}$.

From Lemma 5.2.5, it is obvious that the number of triangles of H is equal to that of F. Let M be the number of triangles of $K(t_1 \times n, t_2 \times (n+1))$ and $N_A(i,j)$ the number of triangles in $H^{-K_{1,s}}(i,j)$, where $i, j = n, n+1$. Then the following results can be obtained easily.

$$\begin{aligned}
&N_A(n,n) = M - s((t_1-2)n + t_2(n+1)),\\
&N_A(n+1,n+1) = M - s(t_1 n + (t_2-2)(n+1)),\\
&N_A(n,n+1) = N_A(n+1,n) = M - s((t_1-1)n + (t_2-1)(n+1)).
\end{aligned} \tag{6.37}$$

By (6.37), we see that $N_A(H) = N_A(F)$ if and only if $H, F \in \{H^{-K_{1,s}}(n, n+1), H^{-K_{1,s}}(n+1,n)\}$. Noticing that

$$H^{-K_{1,s}}(n,n+1) = K^{-K_{1,s}}(n,n+1) + K((t_1-1)\times n, (t_2-1)\times(n+1))$$

and

$$H^{-K_{1,s}}(n+1,n) = K^{-K_{1,s}}(n+1,n) + K((t_1-1)\times n, (t_2-1)\times(n+1)),$$

by Lemma 5.2.1 we have that

$$\sigma(H^{-K_{1,s}}(n,n+1),x) = \sigma(K^{-K_{1,s}}(n,n+1),x)Q(x)$$

and

$$\sigma(H^{-K_{1,s}}(n+1,n),x) = \sigma(K^{-K_{1,s}}(n+1,n),x)Q(x),$$

where $Q(x) = \sigma(K((t_1-1)\times n, (t_2-1)\times(n+1)),x)$.

Since $n \geq s+2$, it follows, from Lemma 6.2.1, that

$$\sigma(K^{-K_{1,s}}(n,n+1),x) \neq \sigma(K^{-K_{1,s}}(n+1,n),x),$$

which implies that

$$P(H^{-K_{1,s}}(n, n+1), \lambda) \neq P(H^{-K_{1,s}}(n+1, n), \lambda).$$

Hence $P(G_1, \lambda) \neq P(G_2, \lambda)$ for G_1 and G_2 in $\mathcal{H}^{-K_{1,s}}$. $\square$

Let $K(n, n, (t-2)\times(n+1))$ has t partition sets A_i such that $|A_1| = |A_2| = n$ and $|A_i| = n+1$ for $3 \leq i \leq t$.

Theorem 6.6.4. *Let $G = K(n, n, (t-2) \times (n+1))$. If $s \geq 1$ and $n \geq s+2$, then $K_{1,2}^{-sK_2}(n, n, (t-2) \times (n+1))$ is χ-unique.*

Proof. Suppose that $H \sim K_{1,2}^{-sK_2}(n, n, (t-2) \times (n+1))$. From Theorem 6.6.2 and Lemma 5.2.7, we have that $H \in \mathcal{G}^{-s}(n, n, (t-2) \times (n+1))$ and $\alpha_{t+1}(H) = s$. Next we consider the number of triangles of H. Without loss of generality, assume that $S \subset E(G)$ and $H = G - S$. Let $e \in S$ and let $N_A(e)$ be the number of triangles in G containing the edge e. Then one can see that $N_A(e) \leq (t-2)(n+1)$ and that equality holds if and only if each $e \in S$ is an edge of the subgraph $K(A_1, A_2)$ of G. So,

$$N_A(H) \geq N_A(G) - s(t-2)(n+1),$$

where equality holds if and only if each $e \in S$ is an edge of the subgraph $K(A_1, A_2)$ of G.

Note that $N_A(H) = N_A(G) - s(t-2)(n+1)$ and $\alpha_{t+1}(H) = s$. By Lemma 5.2.7, we know that $H = K_{1,2}^{-sK_2}(n, n, (t-2) \times (n+1))$. $\square$

Remarks

In this chapter we investigated the chromaticity of general multipartite graphs. In Section 6.3, we study the chromaticity of the tripartite graphs obtained from a complete bipartite graph by adding some edges between vertices of one of the partition sets in the complete bipartite graph. We found all chromatic equivalence classes of graphs $K^{+s}(n, n)$ and a necessary and sufficient condition for $K^{+s}(n,n)$ to be chromatically unique, see Theorems 6.3.1 and 6.3.2, where $n \geq s+2$ and the subgraphs induced by the s edges are bipartite graphs. In Sections 6.4, 6.5 and 6.6, by employing the same method, we investigated the chromaticity of the tripartite graphs obtained from a complete tripartite graph by deleting some edges, the 4-partite graphs obtained from a complete 4-partite graph by deleting some edges and the t-partite graphs obtained from the complete t-partite graph $K(n, n, \cdots, n, n+1, n+1, \cdots, n+1)$ by deleting some edges, respectively. Some parallel results were obtained. First, we gave some lower bounds of three similar functions, see Theorems 6.4.1, 6.5.1 and 6.6.1. Then, by using these theorems and comparing the numbers of $(t+1)$-independent partitions of t-partite graph with those of their corresponding chromatically equivalent graphs, we obtained some sufficient condition for three classes to be χ-closed, see Theorems 6.4.2, 6.5.2 and 6.6.2. Finally, we studied the chromatic uniqueness of graphs with the maximum number of $(t+1)$-independent partitions or with the minimum number of $(t+1)$-independent partitions in these three classes and found some chromatically unique graphs, see Theorems 6.4.3, 6.4.4, 6.5.3, 6.5.4, 6.6.3 and 6.6.4, by considering the number of triangles of a graph and those of its chromatically equivalent graphs. However, it is difficult to get new results on chromaticity of multipartite graphs, by applying the above method.

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Notation

$V(G)$:	vertex set of G
$E(G)$:	edge set of G
$p(G)$:	the number of vertices of G
$q(G)$:	the number of edges of G
$\overline{G}$:	complement of G
$N_G(v)$:	the set of vertices of G adjacent to v
$d_G(v)$ or $d(v)$:	the degree of vertex v
$N_G(e)$:	$N_G(v_1) \cup N_G(v_2) - \{v_1, v_2\}$, where $e = v_1v_2$
$d_G(e)$:	$\lvert N_G(e) \rvert$
$N_A(G)$:	the number of triangles of G
$\eta(G)$:	the edge density of G
$P(G, \lambda)$:	chromatic polynomial of G
$\sigma(G, x)$ or $\sigma(G)$:	σ-polynomial of G
$h(G, x)$ or $h(G)$:	adjoint polynomial of G
$t(G)$:	the lowest term of $h(G, x)$
$\ell(G)$:	the degree of $t(G)$
$h_1(G, x)$ or $h_1(G)$:	polynomial with a nonzero constant term such that $h(G, x) = x^{\ell(G)} h_1(G, x)$
$f(G, x)$:	characteristic polynomial of G
$\rho(G)$:	the maximum roots of $f(G, x)$
$\beta(G)$:	the minimum real roots of $h(G, x)$
$\alpha(G, k)$:	the number of k-independent partitions of G

P_n:	path with n vertices, $\mathcal{P} = \{P_n \| n \geq 2\}$
C_n:	cycle with n vertices, $\mathcal{C} = \{C_n \| n \geq 3\}$
$K_{1,n-1}$:	star with n vertices
K_n:	complete graph with n vertices
O_n:	$\overline{K_n}$
$K(n_1, n_2, \cdots, n_t)$:	complete t-partite graph
$G - S$:	the graph obtained by deleting all edges in S from G
$G + S$:	the graph obtained by adding all edges in S to G that are not already present
$\alpha_r(G - S)$:	$\alpha(G - S, r) - \alpha(G, r)$

For two graphs G and H,

$G \cong H$:	G is isomorphic to H
$G = H$:	$G \cong H$
$G \sim H$:	$P(G, \lambda) = P(H, \lambda)$
$G \sim_h H$:	$h(G, x) = h(H, x)$
$[G]$:	$\{H \| H \sim G\}$
$[G]_h$:	$\{H \| H \sim_h G\}$
$G \cup H$:	the disjoint union of G and H
$G + H$:	the join graph of G and H

For two polynomials $f(x)$ and $g(x)$ in x,

$(f(x), g(x))$:	the greatest common factor of $f(x)$ and $g(x)$
$g(x) \| f(x)$:	$g(x)$ divides $f(x)$
$g(x) \nmid f(x)$:	$g(x)$ does not divide $f(x)$
$\partial f(x)$:	the degree of $f(x)$.

For a real number a,

$\lfloor a \rfloor$:	the largest integer smaller than or equal to a
$\lceil a \rceil$:	the smallest integer larger than or equal to a

For two sets A and B,

$A \backslash B$: the set obtained from A by deleting the elements in B.

Index

Summary

In this thesis, our main aim is to study the algebraic properties of adjoint polynomials and the chromaticity of some classes of graphs. In the first part, Chapters 2 and 3, we concentrate on algebraic properties and roots of adjoint polynomials. In the second part, Chapters 4, 5 and 6, by applying the results of the first part we investigate the chromaticity of some classes of graphs, including dense graphs, complete multipartite graphs and general multipartite graphs.

In Chapter 1 we introduce some basic definitions and terminology and give an overview of our main results, together with some connections with older results.

In Chapter 2 we investigate the recursive relations and divisibility of adjoint polynomials of some families of graphs. As an application of the recursive relations of adjoint polynomials, some new uniquely colorable graphs are obtained.

By using the results of Chapter 2, in Chapter 3 we study the minimum real roots of adjoint polynomials and determine some classes of graph with complex roots. We first give some basic equalities and inequalities on the minimum real roots of adjoint polynomials of some graphs. Then all connected graphs such that the minimum real roots of their adjoint polynomial belong to the interval $[-4, 0]$ and to the interval $[-(2+\sqrt{5}), -4)$ are determined. In the last section of this chapter, we give a way to construct graphs such that their σ-polynomials have at least one complex root. Moreover we solve a problem posed in 1994 by Brenti, Royle and Wagner in Canadian Journal of Mathematics.

Our main goal in Chapter 4 is to study the chromaticity of some dense graphs, by using results in the Chapters 2 and 3. We establish a necessary

and sufficient condition of chromatic uniqueness of a dense graph such that its minimum degree is greater than or equal to the number of vertices minus 3. A necessary and sufficient condition for two graphs H and G with the minimum real roots greater than or equal to -4 to be adjointly equivalent is obtained, too. Two conjectures proposed in 2002 by Dong, Teo, Little and Hendy in Discrete Mathematics are solved. In the last three sections of this chapter, we obtain new results on the adjointly uniqueness of graphs.

In Chapter 5 we turn our attention to the chromaticity of complete multipartite graphs. First, by using some results of the minimum real roots of adjoint polynomials we show that the complete t-partite graph $K(n-k, n, n, \cdots, n)$ is χ-unique, for all $k \geq 2$, $n \geq k+2$ and $t \geq 3$. Then we give some sufficient conditions for complete multipartite graphs to be chromatically unique. Furthermore we solve a conjecture and a problem proposed in 1990 by Koh and Teo in Graphs and Combinatorics.

As a natural generalization of Chapter 5, in the last chapter we study the chromaticity of general multipartite graphs. We investigate the chromaticity of the tripartite graphs obtained from a complete bipartite graph by adding some edges between vertices of one of the partition sets in the complete bipartite graph and of the tripartite graphs obtained from a complete tripartite graph by deleting some edges. In the last two sections, we study the chromaticity of 4-partite graphs and of some t-partite graphs, where $t \geq 5$, and obtain some new results.

Curriculum Vitae

Haixing Zhao was born on May 14, 1969, in Huangzhong County of Qinghai Province, P.R.China. From 1977 to 1987 he attended primary and middle school in his hometown. In September 1987, he started to study pure mathematics at Qinghai Normal University in Xining. After receiving his Bachelor's degree in July 1991, he had been teaching at fifth middle school of Delingha in Qinghai Province until August 1995. After this, he became a graduate student at Qinghai Normal University. He specialized in pure mathematics, and completed his Master's degree thesis, entitled "A necessary and sufficient condition of irreducible path and chromatic uniqueness of complements of L-shaped graphs", under the supervision of Professor Ruying Liu. In June 1998, he received his Master's degree and began his job as a teacher at Qinghai Normal University. He has been teaching computer science and discrete mathematics and doing research on the Reliability of Networks and Graph Theory.

In September 2002, he started as a Ph.D. student under the supervision of Prof. Dr. Kees Hoede at the University of Twente, and Prof. Dr. Xueliang Li from the Northwestern Polytechnical University in Xi'an. The main results of the research on graph theory during the past three years form the content of the thesis that is now in the hands of the reader.